Implied Ambiguity: Mean-Variance Efficiency and Pricing Errors

Chiaki Hara* Toshiki Honda†

October 9, 2018

Abstract

We study the optimal portfolio choice problem for an ambiguity-averse investor having a utility function of the form of Klibanoff, Marinacci, and Mukerji (2005) and Maccheroni, Marinacci, and Ruffino (2013). We identify necessary and sufficient conditions for a given portfolio to be optimal for some ambiguity-averse investor. We also show that the smallest ambiguity aversion coefficient

*Some results of this paper were included in our previous papers entitled "Asset Demand and Ambiguity Aversio" and "Mutual Fund Theorem for the Ambiguity-Averse Investors and the Optimality of the Market Portfolio," which had been circulated since 2013. The research was funded by "Economic Analysis of Intergenerational Issues: Searching for Further Development," a Grant-in-Aid for Specially Promoted Research (grant number 22000001) from Japan's Ministry of Education, Culture, Sports, Science and Technology; "Equilibrium Analysis on Financial Markets with Transaction costs," a Grant-in-Aid (A) (grant number 25245046) from the Japan Society for the Promotion of Science; "Capital Asset Pricing Model of Ambiguity-Averse Investors," from Nomura Fundation; and "Effect of the Change in Trust Act on 2006 on the Efficacy of the Beneficiaries' Decision Making," from Trust Companies Association of Japan. Hara's email address is `hara.chiaki.7x@kyoto-u.ac.jp`.

†The paper is part of the academic project funded by the Grant-in-Aid (grant number 25285097) from Japan's Ministry of Education, Culture, Sports, Science and Technology. Honda's email address is `thonda@hub.hit-u.ac.jp`. We are both grateful to Robert Anderson, Darrell Duffie, Simon Grant, Youichiro Higashi, Mayumi Horie, Atsushi Kajii, Peter Klibanoff, Nobuo Koida, Christoph Kuzmics, Sujoy Mukerji, Frank Riedel, Shuhei Takahashi, Jean-Marc Tallon, Takashi Ui, Ryoko Wada, Katsuhito Wakai, and participants of our presentations at Kyoto University, Hitotsubashi University, Meiji University, Yokohama National University, the 2013 Annual Meeting of the Nippon Finance Association, the Swiss-Kyoto Symposium, the 2013 Asian Meeting of the Econometric Society, the 2014 Annual Meeting of the Korean Association of Financial Engineering, the ZiF Research Group Workshop on Knightian Uncertainty in Strategic Interactions and Markets, the 2015 Annual Meeting of the Asian Finance Association, the 2016 Winter Workshop on Operations Research, Finance and Economics, the Fourth and Fifth Asian Quantitative Finance Conferences, Sookmyung Mathematical Finance Conference, and International Conference on Mathematical Finance Conference at CAMP.

for the optimality of the given portfolio, which we term the implied ambiguity of the portfolio, is decreasing with respect to its Sharpe ratio. This relation can also be expressed in terms of the size of the pricing errors when the asset returns are regressed on the return of the portfolio. A numerical analysis is provided to find the ambiguity aversion implied by the U.S. equity market data.

JEL Classification Codes: D81, D91, G11, G12.

Keywords: Ambiguity aversion, optimal portfolio, Sharpe ratio, beta, alpha, mutual fund theorem.

1 Introduction

Markowitz (1952), Tobin (1958, Section 3.6), Sharpe (1964, Section II), and Lintner (1965, Sections I to III) laid the foundation of modern portfolio theory. They assumed that investors care about both the mean and variance of portfolio returns. When, in addition, a riskless asset is traded and both risky and riskless assets can be sold short without transaction costs, they established the mutual fund theorem: there is a common vector of fractions of investment into risky assets that is optimally held by all investors. The difference in their degrees of risk aversion presents itself as the difference in the ratio at which their wealth is split between risky assets and riskless asset. A more risk-averse investor invests less into risky assets, but, of the total amount invested in risky assets, the fractions over risky assets are the same as those for a less risk-averse investor.

The portfolio of risky assets held in the optimal fractions is called the tangency portfolio, which is illustrated in Figure 1. Inside the curve that is convex to the left are the pairs of means (measured along the vertical axis) and standard deviations (measured along the horizontal axis) of the returns of all portfolios that consist of risky assets. The curve represents the smallest variance (or, equivalently, the smallest standard deviation) of portfolio returns that can be attained while maintaining each level of the mean. The return on the riskless asset is given as a point on the vertical axis, because its standard deviation is zero. The line going through the riskelss return is tangent to the mean-variance efficiency frontier at the return of the tangency portfolio. It maximizes the slope of the line connecting the riskless return and the return of any portfolio of risky assets. The slope is equal to the ratio of its expected excess return to standard deviation and known as the Sharpe ratio of the portfolio.

Although the allocation of investment among risky assets should be independent of investors' risk aversion according to the mutual fund theorem, such a simple rule of investment is rarely observed or recommended. Canner, Mankiw, and Weil (1997) showed that the popular advice on portfolio selection that they surveyed all exhibit the systematic pattern of increasing investment in the more risky asset class (such as stocks) and decreasing investment in the less risky one (such as government bonds), as the investor's risk aversion decreases. Much more recently, some robo-advisors, such as Wealthfront, construct a menu of portfolios, each targeted to a particular level of risk aversion that they infer from the clients' responses to the questionnaires; and as the risk aversion decreases, the portfolios put less weights on less risky asset classes (consisting of, among others, government and corporate bonds in the US) to more risky ones (consisting of, among others, stocks). These suggested portfolios contradict the mutual fund theorem. Instead, they should form just a single portfolio of fixed fractions of investment into various asset classes and advise their clients how much to invest in it according to their risk aversion.

Albeit involving a process of adding up portfolios and finding an equilibrium, a more pronounced, and more important, example of contradiction can be found in the market portfolio in the Captial Asset Pricing Model (CAPM) developed by Sharpe (1964, Section III) and Lintner (1965, Section IV). If all investors hold scalar multiples of the tangency portfolio, then the sum of their portfolios is also a scalar multiple of the tangency portfolio. The sum of their portfolios is, at equilibrium, a scalar multiple of the market portfolio, in which the investment is allocated in proportion to the market capitalizations. Thus, the Sharpe ratio of the market portfolio must be equal to that of the tangency portfolio.

A simple numerical example in Tables 1 and 2, based on the data of the so called FF6 portfolios in the U.S. equity markets from Ken French's website, may help us to see the empirical failure of the mutual fund theorem. The FF6 portfolios are formed by sorting out traded stocks in terms of the market capitalizations, which is either Small or Big, and ratio of the book equity to the market equity (B/M), which is either Low, Neutral, or High, and named the SL, SN, SH, BL, BN, and BH portfolios. They make up a partially aggregated description of the stock market performance, less aggregated than the value-weighted portfolio of all available stocks, that allows us to derive the tangent and market portfolios without suffering from the curse of dimensionality.

The sample means, variances, and covariances of the FF6 portfolios for the period of August 1926 to December 2017 are reported in Table 1. Table 2 reports the tangency portfolio based on the data of Table 1 and, as a proxy of the market portfolio, the value-weighted portfolio of FF6 portfolios, which is computed as the time-series average of market capitalization weights of FF6 portfoios. The Sharpe ratio of the tangency portfolio is equal to 0.22, while the Sharpe ratio of the value-weighted portfolio is equal to 0.13, which is just about sixty percent of the Sharpe ratio of the tangency portfolio. The value-weighted portfolio, by its definition, holds long all FF6 portfolios, and invests only two to three percent in each of the SL, SN, and SH portfolios. The tangency portfolio, on the other hand, exploits the fact that the SN portfolio has a higher expected return and the smaller variance than the SL portfolio by containing a large long position in the SN portfolio, which is financed by a large short position in the SL portfolio.

To reconcile the theory with the reality, we use ambiguity-averse utility functions initiated by Klibanoff, Marinacci, and Mukerji (2005) (hereafter KMM) and further explored by Maccheroni, Marinacci, and Ruffino (2013) (hereafter MMR). Let us emphasize at the outset that we are concerned with the optimal composition of risky assets, rather than how much of the total wealth is split between the risky assets on the one hand and the riskless asset on the other. The latter issue is closely related to the equity premium puzzle of Mehra and Prescott (1985), for example, but we do not take it up in this paper because, as we will see in Section 2.2, the optimal portfolio in our model depends neither on the total wealth level nor on the coefficient of relative ambiguity aversion.

The feature of KMM utility functions that differentiates them from other ambiguity-averse utility functions, such as those of Gilboa and Schmeidler (1985), is that each KMM utility function has a probability distribution on a set of distributions of risky asset returns; and that the aversion to the randomness in the distributions of risky asset returns and the aversion to the randomness in risky asset returns that would still remain once the distribution is known are different. The probability distribution on a set of distributions of risky asset returns represents the perception of ambiguity, which KMM termed as the second-order belief. They also introduced the ambiguity aversion coefficient, which measures, in the same way as the Arrow and Pratt' risk aversion coefficient does, how much more averse the investor is to the ambiguity in distributions than he is to the remaining randomness after the ambiguity in distributions is removed.

In this paper, we assume that the second-order belief is concerned with the expected returns of risky assets; it is a (multivariate) normal distribution; and conditional on the expected returns, the risky asset returns are normally distributed. We also assume that both the aversion to the ambiguity in distributions and the aversion to randomness in risky asset returns that still remain after the ambiguity in distributions is removed are represented by negative exponential utility functions. The resulting model is an ambiguity-inclusive CARA-normal model; and that the resulting utility functions are precisely those considered by MMR.

Theorem 2 and Theorem 3 in Section 4 show, first, that a portfolio of risky assets is optimal for some KMM utility function if and only if its expected excess return is strictly positive, or, equivalently, its return has a strictly positive Sharpe ratio, however low it may be. Second, for each portfolio with a strictly positive expected excess return or Sharpe ratio, they determines the range of the coefficients of absolute risk aversion with which the given portfolio is optimal. Third, for each coefficient of risk aversion in this range, they identify the class of pairs of second-order beliefs and ambiguity aversion coefficients with which the given portfolio is optimal. In particular, for each coefficient of absolute risk aversion, it gives the smallest ambiguity aversion coefficient with which the given portfolio is optimal.

Note that the smallest ambiguity aversion coefficient that Theorem 2 and Theorem 3 identify is subject to the choice of the coefficient of absolute risk aversion. By varying the coefficient of absolute risk aversion, we can find the smallest ambiguity aversion coefficient with which the given portfolio is optimal. We refer to the smallest ambiguity aversion coefficient as the implied ambiguity of the portfolio. The most intriguing result of this paper, Theorem 4, shows that the implied ambiguity is a strictly decreasing function of the Sharpe ratio. The function is zero where the Sharpe ratio is equal to the Sharpe ratio of the tangency portfolio, which is optimal for an ambiguity-neutral investor. As the Sharpe ratio goes down to zero, the implied ambiguity diverges to infinity, meaning that the portfolio is optimal for an extremely ambiguity-averse investor.

Let's illustrate the formula between the Sharpe ratio and the implied ambiguity in the space of mean and standard deviation of portfolio returns. As can be seen on Figure 1, the return of the riskless asset is a point on the vertical axis, and the line going through the return of the riskless asset is tangent to the mean-variance efficiency frontier at the return of the tangency portfolio, as it attains the highest Sharpe ratio. Thus, if you plot the return of any other portfolio, then the line going through it and the riskless asset must necessarily be less steep than the tangential line. This slope determines the implied ambiguity of the portfolio.

We also argue that our result can shed a new light on asset pricing models and their pricing errors, that is, associated alphas. While we saw the empirical failure of the mutual fund theorem as a sign of the failure of the CAPM, the statistical test of the CAPM often takes the form of determining whether there are nonzero alphas, when the asset returns are regressed on a proxy of the market portfolio. The CAPM is rejected whenever the hypothesis of zero alphas is rejected; and this has often been the case in the literature. Gibbons, Ross, and Shanken (1989, Section 3) used the Hotelling's T^2 statistic to test the hypothesis of zero alphas. They also showed that the Sharpe ratio of a portfolio is a decreasing function of the size of alphas when the asset returns are regressed on the return of the portfolio. Based on this result, we show in Corollary 1 that the implied ambiguity is an increasing function of the size of alphas. This corollary is valuable, because it allows us to find the ambiguity aversion at work in asset markets without relying on other data such as experimental findings in laboratories.

To establish these results (Theorem 2, Theorem 3, and Theorem 4 in Section 4), we need to gain a deeper understanding on the optimal portfolios for KMM utility functions. In our first result (Theorem 1 in Section 3), we show that in our ambiguity-inclusive CARA-normal setup, the mutual fund theorem no longer holds, and we characterize the portfolios, which we call the the basis portfolios following Fama (1996), that constitute the building blocks of ambiguity-averse investors' optimal portfolios. The theorem is rich in implications and, among other things, it implies that once a second-order belief is fixed, the composition of risky assets in the optimal portfolio depends only on the ambiguity aversion coefficient, and that the coefficient of absolute risk aversion only affects the ratio at which the total wealth is split between the risky assets and the riskless asset, just as the standard mutual fund theorem claims. We also give a sufficient condition under which just one or two basis portfolios can span the ambiguity-averse investor's optimal portfolio, regardless of his ambiguity aversion coefficient, in which case Theorem 1 may legitimately be called a generalized mutual fund theorem.

In order to obtain quantitative implications from our theoretical analysis, we use U.S. equity market data in Section 6. From the estimates of mean and covariance of equity returns, we numerically find the implied ambiguity of some representative portfolios and the associated second-order beliefs. Through the numerical analysis, we find that the results obtained in our numerical example is robust to the choice of key parameters such as risk aversion coefficients and the weights in the market portfolio.

As for the literature, the contributions that are theoretical in nature and most relevant to this paper have already been mentioned. There is now a growing number of experimental and empirical papers that dealt with ambiguity aversion.

Ahn, Choi, Gale, and Kariv (2014) and Attanasi, Gollier, Montesano, and Pace (2014) inferred ambiguity aversion from laboratory experiments on portfolio selection. Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010) did the same by letting nearly thirty subjects to trade state-contingent consumptions (Arrow assets) without telling them the probabilities for some (but not all) states to occur. Ahn, Choi, Gale, and Kariv (2014) and Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010) concluded that many subjects are ambiguity-averse, and that they appear to have utility functions of Gilboa and Schmeidler (1989) but not of KMM, because they tend to choose unambiguous (purely risky) consumption plans, a tendency that is consistent with utility functions of Gilboa and Schmeidler (1989) but not with those of KMM. Note, however, that according to Proposition 3 of KMM, the KMM utility functions represent approximately the same preference relations as the utility functions of Gilboa and Schmeidler (1989) when the ambiguity aversion coefficient becomes large without bounds. Hence, these experimental results could be interpreted as saying that the subjects have KMM utility functions with extremely high ambiguity aversion coefficients.

Collard, Mukerji, Sheppard, and Tallon (2018) found the ambiguity aversion coefficients, and the countercyclical nature thereof, using the U.S. equity market data. Our framework is static while theirs is dynamic, but we deal with multiple stocks while they deal only with a stock market index (CRSP value-weighted index). In particular, while we are interested in the optimal composition of stocks for ambiguity-averse investors, they are more interested in to what extent the equity premium (as measured by the single stock market index) can be attributed to ambiguity aversion.

Thimme and Völkert (2015) used the consumption and asset market data to estimate the ambiguity aversion coefficient of the KMM utility function using the generalized method of moments. Their results and ours are not directly comparable, because they used a different KMM utility function (which exhibits constant relative, rather than absolute, risk aversion) and conducted a full econometric analysis (while our analysis is, so to speak, a quantitative theoretical exercise). Nonetheless, in the point estimates at which the two papers share the same functional form that represents ambiguity aversion, and the implied ambiguity of the market portfolio found in this paper is, in fact, close to the smallest of their point estimate. The two papers are different in that they fixed the source of ambiguity (which is assumed to be represented by the riskless return and the price-dividend ratio) at the outset of their analysis, while we derived it through the minimization of the ambiguity aversion coefficient that rationalizes the market portfolio.

The rest of this paper is organized as follows. Section 2 sets up the model and give some preliminary results. Section 3 characterizes optimal portfolios and basis portfolios. Section 4 provides necessary and sufficient conditions for a portfolio is optimal for some ambiguity-averse investor, and derive the relation between the implied ambiguity and the Sharpe ratio. Section 5 recasts the results of Section 4 to derive implications on asset pricing. Section 6 obtains the implied ambiguity, second-order beliefs and associated ambiguity measures, from the U.S. equity market data. Section 7 concludes and suggests directions of future research. All proofs and most lemmas are given in the appendix.

2 Setup

2.1 Formulation

The setup of this paper is essentially a special case of those of KMM and MMR and especially close to that of Section 6 of MMR, but we lay it out in a manner that is more suitable to accommodate the additional parametric assumptions. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let M be a random vector defined on Ω. We will later see that M is the expected return of assets and represents the ambiguity in the mind of investors.

For each $\theta > 0$, define $u_\theta : \boldsymbol{R} \to \boldsymbol{R}$ by letting $u_\theta(z) = -\exp(-\theta z)$ for every $z \in \boldsymbol{R}$. This felicity function exhibits constant absolute risk aversion (CARA) and its coefficient is equal to θ. For each $\gamma > 0$ and each $\theta > 0$, define a utility function $U_{\gamma,\theta}$ over a set of random variables Z on Ω by

$$U_{\gamma,\theta}(Z) = E\left[u_\gamma\left(u_\theta^{-1}\left(E\left[u_\theta(Z) \,|\, M\right]\right)\right)\right]. \tag{1}$$

The investors who have a utility function $U_{\gamma,\theta}$ first evaluate the conditional expectation $E\left[u_\theta(Z) \,|\, M\right]$ given M. Then they transform it into the certainty equivalent. Finally they compute expected utility of those certainty equivalent with respect to M using the felicity function u_γ.

We write $\eta = \gamma/\theta - 1$ and call it the coefficient of relative ambiguity aversion. If we write $\varphi_{\gamma,\theta} = u_\gamma \circ u_\theta^{-1}$, then $U_{\gamma,\theta}(Z) = E\left[\varphi_{\gamma,\theta}\left(E\left[u_\theta(Z) \,|\, M\right]\right)\right]$. Since

$$-\frac{\varphi''_{\gamma,\theta}(z)(-z)}{\varphi'_{\gamma,\theta}(z)} = \eta \tag{2}$$

for every $z < 0$, η is equal to the elasticity of the marginal utility from conditional expectations given M. According to Theorem 2 of KMM, the more concave the function $\varphi_{\gamma,\theta}$ is, the more ambiguity-averse the investor is, and the larger the value of η, the more concave the function $\varphi_{\gamma,\theta}$ is.[1] We say that the investor is ambiguity-neutral if $\gamma = \theta$, that is, $\eta = 0$. In this case, $U_{\gamma,\theta}$ is an expected utility function with CARA coefficient θ. We say that an investor is ambiguity-averse if $\gamma > \theta$ and $\eta > 0$. If $\gamma < \theta$ and $\eta < 0$, then the investor is ambiguity-loving, though we will not pay much attention to this case.

Assume that two types of assets are traded. The first one is composed of N assets whose gross returns are represented by an N-variate random vector R defined on Ω. The second one is the riskless asset whose gross return is deterministic and equal to $R_\mathrm{f} \in \boldsymbol{R}$. We assume also that M is an N-variate random vector, and we take M as the expected value of R. We further assume that M and R are jointly normally distributed, and that $E[M] = E[R]$ and $\mathrm{Cov}[M, R] = \mathrm{Cov}[M, M]$. We can thus write

$$\begin{pmatrix} M \\ R \end{pmatrix} \sim \mathscr{N}\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \Sigma_M & \Sigma_M \\ \Sigma_M & \Sigma_R \end{pmatrix}\right). \tag{3}$$

[1] This coefficient, however, differs from that of KMM in that they defined the ambiguity aversion coefficient as $-\varphi''_{\gamma,\theta}(z)/\varphi'_{\gamma,\theta}(z)$. We opt for this coefficient of relative ambiguity aversion because it is constant (independent of z) and still represents the same ranking of concavity as KMM's definition.

This assumption involves no loss of generality.[2]

Then the conditional return of R given M is normally distributed: $R|M \sim \mathscr{N}\left(M, \Sigma_{R|M}\right)$, where $\Sigma_{R|M} = \Sigma_R - \Sigma_M$. Our utility function, thus, embodies the idea that the investor perceives ambiguity in the expected returns of the risky assets; once he has come to believe that the expected excess return vector is equal to $m \in \boldsymbol{R}^N$, he also believes that the asset returns are distributed according to $\mathscr{N}(m, \Sigma_{R|M})$; and the ambiguity or the model uncertainty is distributed according to $\mathscr{N}(\mu, \Sigma_M)$.

Following KMM, we refer to the matrix Σ_M as the second-order belief. It represents the variability of the expected excess returns of the N assets due to model uncertainty. Of the total covariance matrix Σ_R, the second-order belief Σ_M is the covariance matrix of the randomness in asset returns that can be attributed to model uncertainty, and the remaining covariance matrix, $\Sigma_{R|M}$, represents the randomness in asset returns that still remains even after the model uncertainty is removed. For the ease of exposition, denote by $\mathscr{S}^N$ the set of all $N \times N$ symmetric matrices. Denote by $\mathscr{S}^N_{++}$ the set of all symmetric and positive definite $N \times N$ matrix, and by $\mathscr{S}^N_{+}$ the set of all symmetric and positive semidefinite $N \times N$ matrix. For every $\Sigma_1 \in \mathscr{S}^N$ and every $\Sigma_2 \in \mathscr{S}^N$, we write $\Sigma_1 \geqslant \Sigma_2$ if $\Sigma_1 - \Sigma_2 \in \mathscr{S}^N_{+}$. The ordering $\geqslant$ on $\mathscr{S}^N_{+}$ compares the sizes of ambiguity perceived by the investor and the smallest second-order belief is defined with respect to this ordering. We may write $\Sigma_1 \leqslant \Sigma_2$ when $\Sigma_2 \geqslant \Sigma_1$, and $0 \leqslant \Sigma_1$ to mean $\Sigma_1 \in \Sigma^N_{+}$. Then $0 \leqslant \Sigma_M \leqslant \Sigma_R$ and $0 \leqslant \Sigma_{R|M} \leqslant \Sigma_R$.

We assume that $\Sigma_R \in \mathscr{S}^N_{++}$ but not that $\Sigma_M \in \mathscr{S}^N_{++}$. The assumption of the positive definiteness of Σ_R means that the N assets are not redundant. By not imposing the positive definiteness assumption on Σ_M, we can accommodates the situation where there are a fewer common factors underlying model uncertainty of the expected asset returns.

[2] It can be shown that even if M did not satisfy this assumption (possibly with a dimension greater or smaller than N), some linear transformation of M added by some vector of $\boldsymbol{R}^N$ would satisfy this assumption. Indeed, suppose that F is a K-dimensional random vector and

$$\begin{pmatrix} F \\ R \end{pmatrix} \sim \mathscr{N}\left(\begin{pmatrix} \mu_F \\ \mu_R \end{pmatrix}, \begin{pmatrix} \Sigma_F & \Sigma_{FR} \\ \Sigma_{RF} & \Sigma_R \end{pmatrix}\right)$$

Then there is a $D \in \boldsymbol{R}^{K \times N}$ such that $\Sigma_{RF} = D^\top \Sigma_F$. Define $M = D^\top F + (\mu_R - D^\top \mu_F)$. Then (M, R) has the same distribution as assumed in (3).

2.2 Portfolio Choice Problem

Denote by $(x, y) \in \boldsymbol{R}^N \times \boldsymbol{R}$ a portfolio of these $N+1$ assets, representing the monetary amounts invested in each of these assets. Once the state is realized, the portfolio pays out $x^\top R + yR_\mathrm{f}$. Denote the total wealth to be invested in the $N+1$ assets by $W \in \boldsymbol{R}$. Let $\mathbf{1}$ be the vector in $\boldsymbol{R}^N$ of which the N coordinates are all equal to one. Then the budget constraint on the portfolio $(x, y) \in \boldsymbol{R}^N \times \boldsymbol{R}$ is $\mathbf{1}^\top x + y \leqslant W$. The decision maker's utility maximization problem is given by

$$\begin{aligned} \max_{(x,y)\in\boldsymbol{R}^N\times\boldsymbol{R}} \quad & U_{\gamma,\theta}(x^\top R + yR_\mathrm{f}) \\ \text{subject to} \quad & \mathbf{1}^\top x + y \leqslant W. \end{aligned} \tag{4}$$

Define $V_{\gamma,\theta} : \boldsymbol{R}^N \times \boldsymbol{R} \to \boldsymbol{R}$ by letting

$$V_{\gamma,\theta}(x, y) = \mu^\top x + R_\mathrm{f} y - \frac{1}{2} x^\top \left(\gamma\Sigma_M + \theta\Sigma_{R|M}\right) x \tag{5}$$

for every $(x, y) \in \boldsymbol{R}^N \times \boldsymbol{R}$. Since $\Sigma_{R|M} = \Sigma_R - \Sigma_M$, this can be rewritten as[3]

$$V_{\gamma,\theta}(x, y) = \mu^\top x + R_\mathrm{f} y - \frac{\theta}{2} x^\top \Sigma_R x - \frac{\gamma - \theta}{2} x^\top \Sigma_M x.$$

The following lemma shows that $V_{\gamma,\theta}$ represents the same preference ordering over the portfolios as $U_{\gamma,\theta}$.

Lemma 1 *For every $(x, y) \in \boldsymbol{R}^N \times \boldsymbol{R}$, $U_{\gamma,\theta}(x^\top R + yR_\mathrm{f}) = u_\gamma\left(V_{\gamma,\theta}(x, y)\right)$.*

If (x, y) is a solution to the utility maximization problem (4), then $\mathbf{1}^\top x + y = W$. Hence, by Lemma 1, for every $(x, y) \in \boldsymbol{R}^N \times \boldsymbol{R}$, (x, y) is a solution to (4) if x is a solution to

$$\max_{x\in\boldsymbol{R}^N} \quad V_{\gamma,\theta}(x, W - \mathbf{1}^\top x) \tag{6}$$

and $y = W - \mathbf{1}^\top x$. The first-order condition gives the solution to the problem (4) is that

$$\mu - R_\mathrm{f}\mathbf{1} = (\gamma\Sigma_M + \theta\Sigma_{R|M})x = (\theta\Sigma_R + (\gamma - \theta)\Sigma_M)\, x = \theta\left(\Sigma_R + \eta\Sigma_M\right) x. \tag{7}$$

[3]Thus, it is what MMR calls a robust mean-variance utility function.

Since $\Sigma_R + \eta\Sigma_M \in \mathscr{S}^N_{++}$, this is equivalent to[4]

$$x = \frac{1}{\theta} \left(\Sigma_R + \eta\Sigma_M\right)^{-1} (\mu - R_\mathrm{f}\mathbf{1}). \tag{8}$$

There are two things to note in the optimal portfolio (8).[5] First, it is independent of the total wealth W to be invested in the $N + 1$ assets. Thus, any increase in the total wealth does not affect the money amounts x invested in the N assets, but the money amount $W - \mathbf{1}^\top x$ invested in the riskless asset increases exactly by the same amount as the increment in the total wealth. Second, the fractions $(\mathbf{1}^\top x)^{-1} x$ depend on the coefficient η of relative ambiguity aversion, but not on the coefficient θ of absolute risk aversion. For these reasons, we focus on the composition of risky assets, but not on the allocation of the total wealth between the risky assets and the riskless asset.

2.3 Measure of ambiguity perception

One of the building blocks of the KMM utility function is the second-order belief Σ_M. Since it represents the variances and covariances of the expected asset returns, it would be natural to quantify the ambiguity perceived in the return of portfolio v as its variance $v^\top \Sigma_M v$, or standard deviation $\left(v^\top \Sigma_M v\right)^{1/2}$, with respect to Σ_M. There are, however, multiple second-order beliefs that represent an essentially identical perception of ambiguity. In this subsection, we propose an appropriate measure of ambiguity perceived in asset and portfolio returns that does not suffer from complications arising from this multiplicity.

Note first that, for any two triples (θ, Σ_M, η) and $(\theta', \Sigma'_M, \eta')$, if $\theta' = \theta$ and there is a $\kappa > 0$ such that $\Sigma'_M = \kappa\Sigma_M$ and $\eta' = \kappa^{-1}\eta$, then these two triples give rise to the same optimal portfolio via (8). This fact indicates that the second-order belief Σ_M

[4]The equality (24) of MMR is an equivalent characterization of the optimal portfolio (8).

[5]In our terminology, a portfolio is a vector of money amounts invested in the N assets. It is, thus, any N-dimensional vector. In contrast, in the terminology of Fama (1996, page 448), for example, a portfolio is a vector of the fractions of money amounts invested the N assets in the sum of the money amounts invested in these assets. Its coordinates, thus, add up to one. The difference needs some care when, for example, dealing the beta representation of asset returns. However, the sign of the expected excess return of $x \in \boldsymbol{R}^N$ does not depend on which definition to take. Indeed, the expected excess return of $(x, y) \in \boldsymbol{R}^N \times \boldsymbol{R}$ is equal to $(x^\top \mu + y R_\mathrm{f})/(x^\top \mathbf{1} + y) - R_\mathrm{f}$, but this is equal to $(x^\top \mathbf{1} + y)^{-1} \left(x^\top (\mu - R_\mathrm{f}\mathbf{1})\right)$, and it has the same sign as $x^\top (\mu - R_\mathrm{f}\mathbf{1})$. The Sharpe ratio of portfolio x, $x^\top (\mu - R_\mathrm{f}\mathbf{1})/(x^\top \Sigma_R x)^{1/2}$, is also independent of the choice of definition.

and the coefficient η of relative ambiguity aversion are not completely separable, as Epstein (2010, Section 2.4) pointed out.[6] Second, and more specifically to our setting, the second-order belief Σ_M is also not completely separable from the coefficient θ of absolute risk aversion. That is, for any two triples of (θ, Σ_M, η) and $(\theta', \Sigma'_M, \eta')$, if there is a $\tau \in \boldsymbol{R}$ such that $\Sigma'_M = \Sigma_M + \tau\Sigma_R$, $\theta' = (1+\tau\eta)^{-1}\theta$, and $\eta' = (1+\tau\eta)^{-1}\eta$, then they give rise to the same optimal portfolio via (8). Thus, while scalar multiplication of Σ_M and addition of a scalar multiplication of Σ_R to Σ_M can affect the variance $v^\top\Sigma_M v$, they can affect the optimal portfolio only in the way that can be offset by suitably changing the coefficients of risk and ambiguity aversion. Since ambiguity perception is relevant only if it manifests itself in the optimal portfolio, the second-order belief Σ_M should not directly be used as a measure of ambiguity perception.

We define a measure of ambiguity perception that is independent of scalar multiplication of Σ_M and addition of a scalar multiple of Σ_R by normalizing the second-order belief Σ_M in the following manner. Define

$$\begin{aligned}\overline{\lambda}(\Sigma_M) &= \max_{v\in\boldsymbol{R}^N\setminus\{0\}} \frac{v^\top\Sigma_M v}{v^\top\Sigma_R v},\\ \underline{\lambda}(\Sigma_M) &= \min_{v\in\boldsymbol{R}^N\setminus\{0\}} \frac{v^\top\Sigma_M v}{v^\top\Sigma_R v}.\end{aligned}$$

Then $\underline{\lambda}(\Sigma_M)\Sigma_R \leqslant \Sigma_M \leqslant \overline{\lambda}(\Sigma_M)\Sigma_R$ and, thus, $\underline{\lambda}(\Sigma_M) = \overline{\lambda}(\Sigma_M)$ if and only if Σ_M is a scalar multiple of Σ_R. Define the normalized version of the second-order belief by

$$\Phi(\Sigma_M) = \begin{cases} 0 & \text{if } \Sigma_M \text{ is a scalar multiple of } \Sigma_R,\\ \dfrac{1}{\overline{\lambda}(\Sigma_M) - \underline{\lambda}(\Sigma_M)}\left(\Sigma_M - \underline{\lambda}(\Sigma_M)\Sigma_R\right) & \text{otherwise.}\end{cases}$$

Then $\underline{\lambda}(\Phi(\Sigma_M)) = 0$ and, unless Σ_M is a scalar multiple of Σ_R, $\overline{\lambda}(\Phi(\Sigma_M)) = 1$. Thus, $0 \leqslant \Phi(\Sigma_M) \leqslant \Sigma_R$. Two triples (θ, Σ_M, η) and $(\theta', \Sigma'_M, \eta')$ give rise to the same optimal portfolio via (8) regardless of the expected excess returns $\mu - R_{\mathrm{f}}\mathbf{1}$ if and only if $\theta(\Sigma_R + \eta\Sigma_M) = \theta'(\Sigma_R + \eta'\Sigma'_M)$. Since, under some restrictions on the values of η and η',[7] this is equivalent to $\Phi(\Sigma_M) = \Phi(\Sigma'_M)$, the normalized version of a second-order

---

[6] If $\kappa > 1$, then $\kappa^{-1}\Sigma_M \leqslant \Sigma_R$. But, if $\kappa < 1$, it need not be true that $\kappa^{-1}\Sigma_M \leqslant \Sigma_R$. It is for this reason, we will see later i the proof of Theorem 2, that there is a lower bound, but no upper bound, on the coefficient of relative ambiguity aversion with which a given portfolio is optimal.

[7] This restriction is that $-\left(\overline{\lambda}(\Sigma'_M) - \underline{\lambda}(\Sigma'_M)\right)\eta' < \left(\overline{\lambda}(\Sigma'_M)\underline{\lambda}(\Sigma_M) - \overline{\lambda}(\Sigma_M)\underline{\lambda}(\Sigma'_M)\right)\eta\eta' < \left(\overline{\lambda}(\Sigma_M) - \underline{\lambda}(\Sigma_M)\right)\eta$. It is satisfied for all $\eta > 0$ and $\eta' > 0$ if $\underline{\lambda}(\Sigma_M) = \underline{\lambda}(\Sigma'_M) = 0$.

belief represents the class of all second-order beliefs that give rise to the same optimal portfolio regardless of the expected excess returns.

With the normalized second-order belief, we define the measure of ambiguity perceived in the return of each portfolio $v \in \boldsymbol{R}^N$ by

$$\operatorname{Amb}(v \mid \Sigma_M) \equiv \left(\frac{v^\top \Phi(\Sigma_M) v}{v^\top \Sigma_R v} \right)^{1/2}. \tag{9}$$

Then, this measure satisfies $0 \leqslant \operatorname{Amb}(v \mid \Sigma_M) \leqslant 1$ for every v, $\operatorname{Amb}(v \mid \Sigma_M) = 0$ for some $v \in \boldsymbol{R}^N$, and, unless Σ_M is a scalar multiple of Σ_R, $\operatorname{Amb}(v \mid \Sigma_M) = 1$ for some $v \in \boldsymbol{R}^N$. This measure tells us how large the ambiguity perceived in the return of portfolio v is relative to the total variance of portfolio v.

3 Optimal portfolio for an ambiguity-averse investor

In this section, we show that the optimal portfolio consists of what we will call basis (or spanning) portfolios. To state and prove the theorem, we let $\Sigma_X \in \mathscr{S}^N_{++}$ and $0 \leqslant \Sigma_M \leqslant \Sigma_X$ and write

$$Q \equiv \Sigma_R^{-1} \Sigma_M.$$

Then define $\zeta : (-1, \infty) \to \boldsymbol{R}^N$ by letting

$$\zeta(\eta) = (I + \eta Q)^{-1} \Sigma_R^{-1} (\mu - R_\mathrm{f} \mathbf{1}) \tag{10}$$

for every $\eta \in (-1, \infty)$. Then, for every $(\theta, \eta) \in \boldsymbol{R}_{++} \times (-1, \infty)$, (8) can be rewritten as $x = \theta^{-1} \zeta(\eta)$. The function ζ tells us how investor's portfolio depends on η, while the total covariance matrix Σ_R and the second order belief Σ_M are fixed as they are embedded in the definition of ζ. In the language of KMM on page 1869, therefore, the following theorem measures the pure effect of introducing greater ambiguity aversion into a given economic situation.

Theorem 1 (Optimal portfolio in terms of basis portfolios) *Suppose that $\mu - R_\mathrm{f}\mathbf{1} \neq 0$. Then there are a positive integer K and eigenvectors $v_1, v_2, \ldots, v_K$ of Q*

with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_K$ *such that, for every* $\eta > -1$,

$$\zeta(\eta) = \sum_{k=1}^{K} \frac{1}{1 + \lambda_k \eta} v_k. \tag{11}$$

Theorem 1 shows, first and foremost, that the optimal portfolios are linear combinations of eigenvectors of Q, regardless of the coefficient η of relative ambiguity aversion. Following Fama (1996), we call these eigenvectors basis portfolios. Each basis portfolio v is chosen so that the covariance of v and each of the N assets with respect to the second-order belief Σ_M is simply a scaled down version of the covariance with respect to the total covariance matrix Σ_R. In fact, let v be a basis portfolio, with eigenvalue λ. Then $\Sigma_R v$ is the vector of covariances of returns between the N asset and portfolio v, and $\Sigma_M v$ is the vector of covariances of expected returns between the N assets and portfolio v. Since $Qv = \lambda v$ by definition, $\Sigma_M v = \lambda \Sigma_R v$.

Theorem 1 is particularly useful when the number K of the basis portfolio is small. We now give sufficient conditions, in terms of Σ_R and Σ_M but independent of μ and R_{f}, for there to exist at most two distinct mutual funds that cater for all ambiguity-averse investor. In this case, Theorem 1 can legitimately be called the generalized mutual fund theorem.

Proposition 1 (One or two basis portfolios) *1. If there is a* $\lambda \geqslant 0$ *such that* $\lambda \Sigma_R = \Sigma_M$, *then there is a* $v \in \boldsymbol{R}^N$ *such that, for every* $\eta > -1$,

$$\zeta(\eta) = \frac{1}{1 + \lambda \eta} v.$$

2. *If* $0 < \operatorname{rank} \Sigma_M < N$ *and there is a* $\lambda > 0$ *such that* $\operatorname{rank} \Sigma_M + \operatorname{rank} (\lambda \Sigma_R - \Sigma_M) = N$, *then there are a* $v_{\mathrm{R}} \in \operatorname{Ker} \Sigma_M$ *and a* $v_{\mathrm{A}} \in \operatorname{Ker} (\Sigma_R - \Sigma_M)$ *such that, for every* $\eta > -1$,

$$\zeta(\eta) = v_{\mathrm{R}} + \frac{1}{1 + \lambda \eta} v_{\mathrm{A}}.$$

3. *If* $1 = \operatorname{rank} \Sigma_M < N$, *then there are a* $\lambda > 0$, *a* $v_{\mathrm{R}} \in \operatorname{Ker} \Sigma_M$, *and a* $v_{\mathrm{A}} \in \operatorname{Ker} (\Sigma_R - \Sigma_M)$ *such that, for every* $\eta > -1$,

$$\zeta(\eta) = v_{\mathrm{R}} + \frac{1}{1 + \lambda \eta} v_{\mathrm{A}}.$$

Part 1 gives a sufficient condition for a single mutual fund. Part 2 gives a sufficient condition for two mutual funds. Part 3 is a special case of part 2, which shows that the condition of part 2 is met whenever there is essentially only one source of ambiguity. In part 2 and part 3, the subscripts R and A of v_{R} and v_{A} stand for risk and ambiguity. They do indeed make sense, because the demand for v_{R} does not depend on the coefficient η of ambiguity aversion, while the demand for v_{A} vanishes as η becomes large without bounds.

To know more on the basis portfolio in Theorem 1, we consider the following series of maximization problems:[8]

$$\max_{v \in \boldsymbol{R}^N \setminus \{0\}} \quad \frac{v^\top \Sigma_{R|M} v}{v^\top \Sigma_R v}. \tag{12}$$

That is, we look for a portfolio that maximizes the fraction of the unambiguous part of the variance of the return of the portfolio v. Denote a solution by v_1. Next, let $n \geqslant 2$ and $v_1, v_2, \ldots, v_{n-1}$ be portfolios, and consider the following maximization problem, subject to the constraint that the returns of the portfolios must be uncorrelated with every preceding portfolio v_m with $m \leqslant n-1$:

$$\begin{aligned} &\max_{v \in \boldsymbol{R}^N \setminus \{0\}} && \frac{v^\top \Sigma_{R|M} v}{v^\top \Sigma_R v} \\ &\text{s. t.} && v_m^\top \Sigma_R v = 0 \text{ for every } m \leqslant n-1. \end{aligned} \tag{13}$$

We say that a sequence $(v_1, v_2, \ldots, v_N)$ of portfolios is a sequence of solutions to the sequence of problems (12) and (13) if $v_1, v_2, \ldots, v_N$ are obtained by iteratively solving (12) and (13). The next proposition states the relation between basis portfolios and a sequence of solutions to (12) and (13), and shows properties of basis portfolios.

Proposition 2 (Eigenvalues and eigenvectors of Q) 1. *For every sequence $(v_1, v_2, \ldots, v_N)$ of solutions to the sequence of problems* (12) *and* (13) *and for every n, v_n is an eigenvector of Q and its corresponding eigenvalue λ_n is equal to $\left(v_n^\top \Sigma_M v_n\right) / \left(v_n^\top \Sigma_R v_n\right)$.*

2. *For every sequence $(v_1, v_2, \ldots, v_N)$ of eigenvectors of Q, if $v_m^\top \Sigma_R v_n = 0$ whenever*

[8] Similar maximization problems are used by the statistical factor model applying principal components to the covariance matrix. See Campbell, Lo, and MacKinlay (1997, Section 6.4) for example. Here, in stead of maximizing the return variance, we maximize in (12) the fraction of variance by pure risky part of uncertainty.

$m \neq n$ and the sequence of the corresponding eigenvalues is non-decreasing, then it is a sequence of solutions to the sequence of problems (13)*.*

3. *The eigenvectors of Q that correspond to distinct eigenvalues are orthogonal to each other with respect to Σ_R.*

4. *All eigenvalues of Q belong to the closed unit interval* $[0, 1]$*.*

This proposition states that the eigenvectors of Q can be obtained by iteratively maximizing the fraction of the unambiguous part of the variance of portfolio returns. The returns of these eigenvector portfolio are independent with each other with respect to the covariance matrix Σ_R.

Proposition 2 can be used to give a simpler expression of the measure of ambiguity perception in (9). It tells us that $\underline{\lambda}(\Sigma_M) = \lambda_1$ and $\overline{\lambda}(\Sigma_M) = \lambda_N$. Thus, for every $v \in \boldsymbol{R}^N$,

$$\mathrm{Amb}\,(v \mid \Sigma_M) = \left(\frac{1}{\lambda_N - \lambda_1} \left(\frac{v^\top \Sigma_M v}{v^\top \Sigma_R v} - \lambda_1 \right) \right)^{1/2}.$$

Moreover, v is an eigenvector of Q that corresponds to its smallest eigenvalue λ_1 if and only if $\mathrm{Amb}\,(v \mid \Sigma_M) = 0$, and v is an eigenvector of Q that corresponds to its largest eigenvalue λ_N if and only if $\mathrm{Amb}\,(v \mid \Sigma_M) = 1$.

4 Ambiguity implied by the Sharpe ratio

4.1 Reverse-engineering problem

In section 3, we found the optimal portfolio of a given ambiguity-averse investor. In this section, we study a reverse-engineering problem, that is, for a given portfolio, we ask whether there is an ambiguity-averse investor for whom the portfolio is optimal. The motivation for this study is to fill in the gap between the theory, the mutual fund theorem, and the reality, the systematic deviation from the theorem at both individual and aggregate levels. In doing so, we identify an equivalent condition for a given portfolio to be optimal for some ambiguity-averse investor; show the multiplicity of such investors in case there is one; characterizing all such investors; and find the least ambiguity-averse one of all such investors.

In the subsequent analysis, we will take the total covariance matrix Σ_R, the mean vector μ, the riskless asset return R_{f}, and a portfolio x as given. Then we infer a coefficient θ of absolute risk aversion, a coefficient η of relative ambiguity aversion, and a second-order belief Σ_M for which the portfolio x is optimal. From a mathematical viewpoint, we solve equation (8) for θ, Σ_M, and η, when R_{f}, μ, Σ_R, and x are given. From a decision-theoretic viewpoint, we show how much of the total covariance matrix Σ_R the investor attributes to ambiguity and to what extent the investor is averse to risk and ambiguity if the given portfolio x is optimal for him.

4.2 Necessary and sufficient conditions for the optimality of a portfolio

Suppose that a portfolio x is optimal and (8) holds for an ambiguity-averse investor who has the coefficient θ of absolute risk aversion, the second-order belief Σ_M, and the coefficient η of relative ambiguity aversion. Theorem 2 derives some inequalities that bound the values of θ, η, and Σ_M. These inequalities are necessary conditions for the optimality of portfolio x in terms of the ambiguity-averse investor's attitudes towards risk, attitudes toward ambiguity, and perception of ambiguity.

Theorem 2 (Necessary conditions of an optimal portfolio) *Let $\Sigma_R \in \mathscr{S}^N_{++}$, $\mu \in \boldsymbol{R}^N$, $R_{\mathrm{f}} \in \boldsymbol{R}$, $\theta > 0$, $\Sigma_M \in \mathscr{S}^N_{+}$, $\eta \geqslant 0$, and $x \in \boldsymbol{R}^N$. Suppose that $\Sigma_M \leqslant \Sigma_R$ and (8) holds. Then*

$$(\mu - R_{\mathrm{f}}\mathbf{1})^\top x > 0. \tag{14}$$

Define

$$\bar{\theta} = \frac{(\mu - R_{\mathrm{f}}\mathbf{1})^\top x}{x^\top \Sigma_R x}. \tag{15}$$

Then

$$\theta < \bar{\theta}. \tag{16}$$

Define

$$v^\theta = \frac{1}{\theta}\Sigma_R^{-1}(\mu - R_{\mathrm{f}}\mathbf{1}) - x. \tag{17}$$

Suppose that $v^\theta \neq 0$ *and define*

$$\Sigma_M^\theta = \frac{1}{\left(v^\theta\right)^\top \Sigma_R v^\theta} \left(\Sigma_R v^\theta\right) \left(\Sigma_R v^\theta\right)^\top, \tag{18}$$

$$\eta^\theta = \frac{\left(v^\theta\right)^\top \Sigma_R v^\theta}{x^\top \Sigma_R v^\theta}. \tag{19}$$

Then

$$\eta \Sigma_M \geqslant \eta^\theta \Sigma_M^\theta, \tag{20}$$

$$\eta \geqslant \eta^\theta. \tag{21}$$

The first inequality (14) is the most fundamental one. It states that the expected excess return of portfolio x is strictly positive. Once this condition is met, the second inequality (16) gives an upper bound $\bar{\theta}$ on the coefficient of absolute risk aversion. Finally, (20) gives a lower bound $\eta^\theta \Sigma_M^\theta$ on the second-order belief Σ_M multiplied by the coefficient η of relative ambiguity aversion; and (21) gives a lower bound η^θ on the coefficient η of relative ambiguity aversion alone. Note that the lower bounds of (20) and (21) depend on the coefficient θ of absolute risk aversion.

The bounds in Theorem 2 are also sufficient. Theorem 3 proves, indeed, that the lower bounds in (20) and (21) are attained by some ambiguity-averse investor's second-order belief and coefficient of relative ambiguity aversion.

Theorem 3 (Construction of ambiguity) *Let* $\Sigma_R \in \mathscr{S}_{++}^N$, $\mu \in \boldsymbol{R}^N$, $R_\mathrm{f} \in \boldsymbol{R}$, *and* $x \in \boldsymbol{R}^N$. *Suppose that* (14) *holds. Define* $\bar{\theta}$ *as in* (15). *Let* $\theta > 0$ *and suppose that* (16) *holds. Define* v^θ *as in* (17) *and suppose that* $v^\theta \neq 0$. *Define* Σ_M^θ, *and* η^θ *as in* (18) *and* (19). *Then* $0 \leqslant \Sigma_M^\theta \leqslant \Sigma_R$, $\eta^\theta > 0$, *and*

$$x = \frac{1}{\theta} \left(\Sigma_R + \eta^\theta \Sigma_M^\theta\right)^{-1} (\mu - R_\mathrm{f} \mathbf{1}). \tag{22}$$

Theorem 3 shows that Σ_M^θ can be the second-order belief of some ambiguity-averse investor, because it is positive semidefinite and at most Σ_R; η^θ can be the coefficient of relative ambiguity aversion of some ambiguity-averse investor because it is strictly positive; and the portfolio x is optimal for the investor with Σ_M^θ and η^θ, because (22) holds. Theorem 3 implies that the last two inequalities, (20) and (21), of Theorem 2 constitute a sufficient condition for the portfolio x to be optimal for some ambiguity-averse investor whose coefficient of absolute risk aversion is equal to θ.

It is illustrative to look at v^θ in (17) and Σ_M^θ in (18) of Theorem 2 through the results of the preceding sections. We see that v^θ is defined to fill in the gap between a scalar multiple of the tangency portfolio and a given portfolio x. The second-order belief Σ_M^θ is defined so that the variance of any portfolio v with respect to Σ_M^θ is a function of its covariance with v^θ, because

$$v^\top \Sigma_M^\theta v = \frac{\left(v^\top \Sigma_R v^\theta\right)^2}{(v^\theta)^\top \Sigma_R v^\theta}.$$

The corresponding matrix Q defined in Section 2, $\Sigma_R^{-1}\Sigma_M^\theta$, has rank 1 and a unique strictly positive eigenvalue 1. Part 3 of Proposition 1 is therefore applicable to Σ_M^θ, with v_{A} equal to a scalar multiple of v^θ and λ equal to 1. Hence, for each portfolio $v \in \boldsymbol{R}^N$,

$$\mathrm{Amb}\,(v \mid \Sigma_M^\theta) = \frac{|(v^\theta)^\top \Sigma_R v|}{((v^\theta)^\top \Sigma_R v^\theta)^{1/2}(v^\top \Sigma_R v)^{1/2}}. \tag{23}$$

Thus, the measure of ambiguity perceived in the return of v is equal to the absolute value of the correlation coefficient between the return on the portfolio v and the return on the portfolio v^θ.

4.3 Implied ambiguity and Sharpe ratio

According to Theorem 3, the coefficient η^θ of relative ambiguity aversion is the smallest one with which x is optimal, given a coefficient θ of absolute risk aversion. In this subsection, we consider the problem of minimizing the coefficient η^θ of relative ambiguity aversion by varying θ. Theorem 4 solves this problem and finds the smallest coefficient of relative ambiguity aversion coefficients that is independent of the choice of θ. The coefficient, therefore, represents the smallest deviation from ambiguity-neutral mean-variance utility functions that is necessary for the given portfolio to be optimal.

We should add a caveat to this exercise. We minimize the coefficient η^θ of relative ambiguity aversion, but we do not really find the least ambiguity-averse investors, in the sense of KMM, for whom the given portfolio is optimal. Theorem 2 of KMM showed that an investor is more ambiguity-averse than another if and only if the function $\varphi_{\gamma,\theta}$ is more concave for the first one than for the second, but the theorem is valid only when the two investors share the same risk attitudes. More specifically, suppose that the smallest coefficients η^θ and $\eta^{\theta'}$ of relative ambiguity aversion, corresponding to two different coefficients, θ and θ', of absolute risk aversion, are defined as in Theorem 2 from a portfolio x. We can say that an investor with utility function $U_{(1+\eta)\theta,\theta}$ is more ambiguity-averse than the investor with the utility function $U_{(1+\eta^\theta)\theta,\theta}$ in the sense of KMM, whenever $\eta > \eta^\theta$. But, even when $\eta^{\theta'} > \eta^\theta$, we cannot say that the investor with the utility function $U_{(1+\eta^{\theta'})\theta',\theta'}$ is more ambiguity-averse than the investor with the utility function $U_{(1+\eta^\theta)\theta,\theta}$ in the sense of KMM, because $\theta \neq \theta'$. Nonetheless, we consider the problem of minimizing η^θ by varying θ because, as our Theorem 1 shows, the composition of the optimal portfolio depends on the coefficient η of relative ambiguity aversion and but not on the coefficient θ of absolute risk aversion.

As it turns out, the Sharpe ratio of the optimal portfolio is closely related to the minimized coefficients of relative ambiguity aversion by varying θ. Formally, define $\mathrm{Sr} : \boldsymbol{R}^N \backslash \{0\} \to \boldsymbol{R}$ by

$$\mathrm{Sr}\,(x) = \frac{(\mu - R_{\mathrm{f}}\mathbf{1})^\top x}{(x^\top \Sigma_R x)^{1/2}}.$$

The Sharpe ratio of a portfolio is the expected excess return that the portfolio can attain per unit of standard deviation of its return. It is often considered as a measure of investment efficiency of the portfolio. The function Sr is homogeneous of degree zero and maximized at every scalar multiple of the tangency portfolio, say, $\overline{x} \equiv \Sigma_R^{-1}(\mu - R_{\mathrm{f}}\mathbf{1})$. The maximized value $\mathrm{Sr}\,(\overline{x})$ is equal to $\left((\mu - R_{\mathrm{f}}\mathbf{1})^\top \Sigma_R^{-1}(\mu - R_{\mathrm{f}}\mathbf{1})\right)^{1/2}$. We thus call the ratio $\mathrm{Sr}\,(x)/\,\mathrm{Sr}\,(\overline{x})$ as the normalized Sharpe ratio of portfolio x, which lies on $[-1, 1]$. The following theorem gives the relation between the normalized Sharpe ratio and the minimized coefficients of relative ambiguity aversion by varying θ.

Theorem 4 (Implied ambiguity) *Let $\Sigma_R \in \mathscr{S}^N_{++}$, $\mu \in \boldsymbol{R}^N$, $R_{\mathrm{f}} \in \boldsymbol{R}$, and $x \in \boldsymbol{R}^N$. Suppose that* (14) *holds. Define $\bar{\theta}$ be as in* (15)*. Let $\theta > 0$ and suppose that* (16) *holds. Define η^θ as in* (19)*. Then, as a function of θ defined on $(0, \bar{\theta})$, η^θ is minimized* *when θ is equal to*

$$\theta^* \equiv \frac{\bar{\theta}}{1 + \left(1 - \left(\dfrac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2}} \tag{24}$$

and the minimized value is equal to

$$\eta^{\theta^*} = \frac{2\left(1 - \left(\dfrac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2}}{1 - \left(1 - \left(\dfrac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2}}. \tag{25}$$

Theorem 4 implies that there is a one-to-one correspondence between the normalized Sharpe ratio of a portfolio x and the smallest coefficient of relative ambiguity aversion with which x is optimal. There is also a one-to-one correspondence between the normalized Sharpe ratio and the coefficient of absolute risk aversion that attains the smallest coefficient of relative ambiguity aversion. As the normalized Sharpe ratio decreases from one to zero, the smallest coefficient of relative ambiguity aversion increases strictly from 0 to ∞, and the coefficient of absolute risk aversion decreases strictly from $\bar{\theta}$ to $\bar{\theta}/2$. We refer to the smallest coefficient η^{θ^*} of relative ambiguity aversion as the implied ambiguity of the portfolio x. As the square-root

$$\left(1 - \left(\frac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2} \tag{26}$$

represents the shortfall of the Sharpe ratio of the return of the portfolio x from the highest Sharpe ratio, attainable by the tangency portfolio $\overline{x}$,[9] (25) shows that the implied ambiguity is an increasing function of the shortfall.

As we mentioned in Footnote 6 in Section 2.3, if a portfolio x is optimal for an investor with a coefficient η of relative ambiguity aversion, then it is also optimal for an investor with a coefficient of relative ambiguity aversion that is greater than η. Thus, a portfolio with a positive expected excess return is optimal for some investor

[9] The square-root (26) can be shown to be equal to $\min_\theta (v^\theta)^\top \Sigma_R v^\theta$ divided by $x^\top \Sigma_R x$, where v^θ is defined by (17). As the division is for normalization, it measures the deviation of the portfolio x from the scalar multiples of the tangency portfolio in terms of the variances of the returns.

with a coefficient η of relative ambiguity aversion if and only if

$$\eta \geqslant \frac{2\left(1-\left(\dfrac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2}}{1-\left(1-\left(\dfrac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2}}.$$

This is perhaps the easiest-to-grasp relation between the normalized Sharpe ratio and the coefficient of relative ambiguity aversion.

When choosing among the $(\theta, \Sigma_M^\theta, \eta^\theta)$'s in Theorem 2, minimizing η^θ is not the only "right" criterion. In closing this section, we mention that it is possible to minimize, instead, the largest eigenvalue (which is the only strictly positive eigenvalue) of the aversion-weighted second-order belief $\eta^\theta \Sigma_M^\theta$ by varying θ. This objective function may be considered as more appropriate, because, as mentioned in Subsection 2.3, the optimal portfolio depends on $\eta^\theta \Sigma_M^\theta$, but not separately on η^θ and Σ_M^θ. This minimization can also be solved by Lemma 4 in the appendix.[10]

5 Ambiguity implied by the pricing errors

In Section 4, we related the shortfall of the Sharpe ratio of a given portfolio to the implied ambiguity. Note here that the first-order condition (7) for the optimal portfolio also gives the pricing portfolio (the portfolio whose return replicates the state price density) of the state price beta model in the sense of Duffie (2001, Section F of Chapter 1). But the given portfolio x itself is, in general, not the pricing portfolio of the model because the first-order condition involves ambiguity aversion. In other words, if we regress the excess returns of the asset on the return of the given portfolio x, then we would end up with nonzero pricing errors, known as the alphas. In this section, we relate the size of these pricing errors to the implied ambiguity, and argue that the relation can be used to assess whether a particular coefficient of relative ambiguity aversion may appropriately be used for asset pricing models.

Denote the vector of alphas of the given portfolio x by $\alpha(x)$, then

$$\alpha(x) \equiv (\mu - R_\mathrm{f}\mathbf{1}) - \frac{(\mu - R_\mathrm{f}\mathbf{1})^\top x}{x^\top \Sigma_R x}\Sigma_R x. \tag{27}$$

[10] Take $S = \Sigma^2$ in the lemma.

If x satisfies the first-order condition (7), then, by letting $Q = \Sigma_R^{-1}\Sigma_M$ and multiplying $(x + \eta Qx)^\top$ from left to both sides of (7), we obtain $\alpha(x + \eta Qx) = 0$. That is, the state price beta model is valid when the pricing portfolio is $x + \eta Qx$.

In general, $\alpha(x) \neq 0$. In order to quantify the size of this N-dimensional vector, we use, among other candidates, the norm

$$\left(\alpha(x)^\top \Sigma_R^{-1} \alpha(x)\right)^{1/2}. \tag{28}$$

The choice of the positive definite matrix Σ_R^{-1} is appropriate on several statistical and econometric grounds. First and foremost, the inverse gives the Wald test statistic of the null hypothesis that the alphas are all zero.[11] Shanken (1987) used the inverse of the covariance matrix of the regression residuals, but the two norms based on the two different covariance matrices can be shown to be equal by using generalized inverses. Gibbons, Ross, and Shanken (1989) used the Hotelling's T^2 statistic, which is an increasing function of our norm of alphas, to test the null hypothesis that the alphas are all zero. Hansen and Jagannathan (1997) evaluated the performance of alternative asset pricing models based on the size of errors in the predicted asset prices using the inverse of the matrix of the non-central second moments, rather than the covariance matrix, of asset returns. The difference is, however, inessential, because we look into the expected excess returns of risky assets under the assumption that the riskless asset is correctly priced regardless of the choice of second-order beliefs and coefficients of relative ambiguity aversion.

Let $\underline{x}$ be a portfolio of zero expected excess return, that is, $(\mu - R_\mathrm{f}\mathbf{1})^\top \underline{x} = 0$.[12] Then the norm (28) is maximized when $x = \underline{x}$, the maximized norm $\left(\alpha(\underline{x})^\top \Sigma_R^{-1} \alpha(\underline{x})\right)^{1/2}$ is equal to $\left((\mu - R_\mathrm{f}\mathbf{1})^\top \Sigma_R^{-1} (\mu - R_\mathrm{f}\mathbf{1})\right)^{1/2}$, and the Sharpe ratio $\mathrm{Sr}\left(\underline{x}\right)$ is equal to 0. The norm (28) is minimized when $x = \overline{x}$, the minimized norm $\left(\alpha(\overline{x})^\top \Sigma_R^{-1} \alpha(\overline{x})\right)^{1/2}$ is equal to 0, and the Sharpe ratio $\mathrm{Sr}\left(\overline{x}\right)$ is equal to $\left((\mu - R_\mathrm{f}\mathbf{1})^\top \Sigma_R^{-1} (\mu - R_\mathrm{f}\mathbf{1})\right)^{1/2}$. More generally, the following relation holds between the Sharpe ratio of a portfolio and the norm of alphas when it is taken as the single pricing factor. It is equivalent to equality (7) and (23) of Gibbons, Ross, and Shanken (1989, Section 3) and can be proved via a straightforward calculation.

[11] Campbell, Lo, and MacKinlay (1997, Section 5.3) gives an extensive account on this point.

[12] Since $(\mu - R_\mathrm{f}\mathbf{1})^\top \underline{x} = \left(\Sigma_R^{-1}(\mu - R_\mathrm{f}\mathbf{1})\right)^\top \Sigma_R \underline{x}$, $\underline{x}$ is a zero-beta portfolio against the tangency portfolio.

Lemma 2 (Gibbons, Ross, Shanken (1989)) *For every $x \in \boldsymbol{R}^N \backslash \{0\}$,*

$$(\mathrm{Sr}\ (x))^2 + \alpha(x)^\top \Sigma_R^{-1} \alpha(x) = (\mu - R_\mathrm{f}\mathbf{1})^\top \Sigma_R^{-1} (\mu - R_\mathrm{f}\mathbf{1}).$$

The lemma states that the squared Sharpe ratio of a portfolio and the squared norm of alphas of the N assets, when the beta relation is taken with respect to the portfolio, add up to the squared highest Sharpe ratio attainable by the tangency portfolios. The equality can be rewritten as

$$\left(\frac{\alpha(x)^\top \Sigma_R^{-1} \alpha(x)}{\alpha(\underline{x})^\top \Sigma_R^{-1} \alpha(\underline{x})}\right)^{1/2} = \left(1 - \left(\frac{\mathrm{Sr}\ (x)}{\mathrm{Sr}\ (\overline{x})}\right)^2\right)^{1/2}. \tag{29}$$

Recall that in Section 4, we saw that the right-hand side, which is the same as (26), represents the shortfall of the Shape ratio of the return of the portfolio x from the highest Sharpe ratio, attainable by the tangency portfolio $\overline{x}$. We see here that the normalized norm of alphas, $\left(\alpha(x)^\top \Sigma_R^{-1} \alpha(x) / \alpha(\underline{x})^\top \Sigma_R^{-1} \alpha(\underline{x})\right)^{1/2}$, is equal to the shortfall of the Sharpe ratio.

The following corollary of Theorem 4 can be immediately obtained from Lemma 2. It allows us to relate the implied ambiguity with the alphas of the asset returns.

Corollary 1 *Let $\Sigma_R \in \mathscr{S}_{++}^N$, $\mu \in \boldsymbol{R}^N$, $R_\mathrm{f} \in \boldsymbol{R}$, and $x \in \boldsymbol{R}^N \backslash \{0\}$. Suppose that* (14) *holds. Define $\bar{\theta}$ be as in* (15)*. Let $\theta > 0$ and suppose that* (16) *holds. Define η^θ as in* (19) *and θ^* as in* (24)*. Then*

$$\eta^{\theta^*} = \frac{2\left(\dfrac{\alpha(x)^\top \Sigma_R^{-1} \alpha(x)}{\alpha(\underline{x})^\top \Sigma_R^{-1} \alpha(\underline{x})}\right)^{1/2}}{1 - \left(\dfrac{\alpha(x)^\top \Sigma_R^{-1} \alpha(x)}{\alpha(\underline{x})^\top \Sigma_R^{-1} \alpha(\underline{x})}\right)^{1/2}}.$$

We now argue that Corollary 1 can also enable us to gain some idea on reasonable coefficients of relative ambiguity aversion. To see how this can be done, recall that the state price beta model is valid if the ambiguity-induced term $\eta \Sigma_M x$ is added to the given portfolio x. Suppose, then, that the given portfolio x is the market portfolio and that we have somehow come to believe that the representative investor, who holds the market portfolio by definition, has a coefficient of relative ambiguity aversion equal to 8.[13] Unlike the case of coefficient of relative risk aversion, on which there is a vast literature ever since Mehra and Prescott (1985) raised the equity premium puzzle, we do not really know whether this coefficient is reasonable. Yet, we know that the state price beta model has zero alphas if its pricing factor is $x + \eta Qx$, rather than x itself. Since $8 = (2 \times 0.8)/(1 - 0.8)$, Corollary 1 implies that the unexplained 80% of the expected excess returns with pricing portfolio x can be removed by assuming the coefficient of relative ambiguity aversion equal to 8. Judging from the asset pricing literature, a typical opinion in the profession would probably be that such a high proportion of expected excess returns should not be attributed to ambiguity aversion alone: rather, market incompleteness, transaction costs, and other impediments to efficient risk sharing should contribute to the unexplained 80%. In this sense, the coefficient of relative ambiguity aversion being equal to 8 would be deemed unreasonably high. The lesson from this exercise is that by combining with our general understanding on asset pricing, we can use Corollary 1 to assess if a coefficient of relative ambiguity aversion is reasonable or not.

To appreciate the importance of inferring the coefficients of relative ambiguity aversion from the return data in asset markets, note that it is difficult to infer the coefficients of relative ambiguity aversion from experimental evidences that are applicable to asset pricing. The reason is that state spaces in experiments are intentionally designed to be simple, often mimicking that of the paradox of Ellsberg (1961), while state spaces that describe ambiguity surrounding asset markets would presumably be much more complicated because, according to KMM (page 1856), state spaces would typically describe the inner working of firms and wider market environments. The ambiguity attitudes of subjects in experiments, though potentially informative, may thus be quite different from the ambiguity attitudes of investors in asset markets.[14] On the other hand, the normalized norm of alphas in Corollary 1 can be estimated from observed asset returns without relying on experimental evidences. Hence, it gives us a better sense of which ambiguity aversion coefficients may appropriately be used for asset pricing models.

[13] As we will see in the next section, this is close to the estimate obtained from the U.S. equity market data.

[14] A similar point was made by Epstein (2010, Section 2.4), where he gave an example of choices that cannot be rationalized by any utility function KMM if a decision maker's attitudes towards risk and ambiguity "travel" with him across settings. Another, related, problem is that when the state spaces are different, there is no obvious way to translate the second-order belief in one setting to that in the other.

6 Examples based on the U.S. equity market data

6.1 Data

In this section, we use the U.S. equity market data to derive quantitative implications from our theoretical analysis in the previous sections. Table 1 shows the sample means, variances, and covariances of the monthly returns of the FF6 portfolios in the U.S. equity markets from July 1926 to December 2017, obtained from Ken French's website. Throughout this section, we take the risless asset return R_{f}, the expected return vector μ, and the total covariance matrix Σ_R to be the the sample means and covariances of the returns in Table 1. In the table, we see that the small equity portfolios tend to have higher average returns than the big equity portfolios. This is known as the small-size effect. The high B/M equity portfolios tend to have higher average returns than the low B/M equity portfolios. Since the stocks with high B/M ratios are called value stocks, and the stocks with low B/M ratios are called growth stocks, this is known as the value effect.

Table 2 reports the allocations, the expected excess returns, the standard deviations, and the Sharpe ratios of the tangency portfolio, the value-weighted portfolio, the equal-weighed portfolio, and the global minimum variance (GMV) portfolio. The tangency and GMV portfolios are calculated from the sample means, variances, and covariances. They are normalized so that the coordinates add up to one. The value-weighted portfolio is the vector of averages of the fractions of market capitalizations of the FF6 portfolios, where the average is taken over the sample period of July 1926 to December 2017. We use this portfolio as a proxy of the market portfolio. The equal-weighted portfolio is the vector of averages of the fractions of the numbers of the firms in the FF6 portfolios, where the average is taken over the same sample period.

Table 2 also reports the alphas, betas, and Sharpe ratios of the FF6 portfolios. The alphas are the pricing errors, not explainable by the betas with the market portfolio. Nonzero alphas have been reported in many empirical studies, such as Black, Jensen, and Scholes (1972) and Fama and French (1992). In our example, the SN, SH, and BH portfolios have positive alphas and the SL portfolio has a negative alpha. The relation between alphas and Sharpe ratios is unclear. In fact, the SN and SH portfolios have almost the same Sharpe ratios, but the SH portfolio has a much larger alpha than the SN portfolio. The portfolios BL, BN, and BH have almost the same Sharpe ratios, but the BH portfolio has a much larger alpha than the other two portfolios.

The tangency portfolio contains large long and short positions. For example, the SN portfolio is bought more than three times the initial wealth to be invested into all stocks, and this is more or less financed by short-selling the SL portfolio. Since, as shown by the sample covariance matrix in Tables 1, the FF6 portfolios are highly positively correlated with each other, the tangency portfolio attains the highest Sharpe ratio of 0.22 by taking full advantage of differences in expected returns among the FF6 portfolios.

The value-weighted portfolio allocates more than ninety percent to the BL, BN, and BH portfolios, which are the portfolios of firms with large market capitalizations. Only a few percents are allocated to the SL, SN, and SH portfolios, which consists of firms with small market capitalizations. The Sharpe ratio of the value-weighted portfolio is equal to 0.13, which is about sixty percent of that of the tangency portfolio.

The equal-weighed portfolio allocates more than twenty percent to each of the SL, SN, and SH portfolios. With the small-size effect, the equal-weighed portfolio has a higher expected return and a larger standard deviation than the value-weighted portfolio. The Sharpe ratio of the equal-weighed portfolio is 0.13, equal to that of the value-weighted portfolio.

The GMV portfolio does not take into account the expected returns but does minimize the standard deviation of the return. Like the value-weighted portfolio, the GMV portfolio also takes large long and short positions. The standard deviation of portfolio is lowest at 4.75. The expected return is 0.86, lower than 1.19 of the value-weighted portfolio. Yet, its Sharpe ratio is 0.12, which is not much lower than that of the value-weighted portfolio, which is 0.13.

6.2 Implied ambiguity and second-order belief

Theorem 2 and Theorem 3 show the equivalent conditions for a portfolio to be optimal for some ambiguity-averse investor. With our data in Table 2, the value-weighted, equal-weighted, and GMV portfolios all satisfy the conditions, and, in this subsection, we numerically find such investor's coefficient of relative ambiguity aversion and second-order belief.

Theorem 2 finds the upper bound $\bar{\theta}$, defined in (15), on the coefficient of absolute risk aversion. Theorem 4 finds the coefficient θ^* of absolute risk aversion that minimizes the coefficient η^θ of relative ambiguity aversion, and the minimized (smallest) coefficient is what we termed the implied ambiguity. Table 3 lists the implied ambiguity η^{θ^*} for the three portfolios in Table 2, with corresponding $\bar{\theta}$ and θ^*. Unfortunately, since these coefficients of absolute risk aversion are subject to the way the portfolios are normalized, and the portfolios we consider are all normalized so that the coordinates add up to one, we cannot infer any information on the investor's risk aversion from these values. Table 3 also reports the normalized Sharpe ratios $\mathrm{Sr}\,(x)/\,\mathrm{Sr}\,(\overline{x})$ and the normalized norm of alphas, $\left(\alpha(x)^\top \Sigma_R^{-1}\alpha(x)/\alpha(\underline{x})^\top \Sigma_R^{-1}\alpha(\underline{x})\right)^{1/2}$, which are related to the coefficient η^{θ^*} of ambiguity aversion in Theorem 4 and Corollary 1. Since they are related with each other via Lemma 2, the lower the normalized Sharpe ratio of the given portfolio, the larger the pricing errors. Indeed, the three portfolios can be ranked in the descending order, as the equal-weighted, value-weighted, and GMV portfolios, with respect to the Sharpe ratios, but this ranking is in the ascending order with respect to the norm of alphas.

Table 4 shows the measures of ambiguity perception of the FF6 portfolios, defined in (23) from the second-order belief, for the three portfolios in Table 2. Thanks to (23), the measure of ambiguity perception of each portfolio can be derived simply from computing its covariance with v^θ, defined in (17). For all of the three portfolios, the measures are ranked, in the ascending order, as the SL, BL, BN, BH, SN, SH portfolios, except, marginally, between the BL and BN portfolios in the case of the equal-weighted portfolio. Here we observe two patterns: The returns of portfolios of small firms are perceived as more ambiguous, except for the SL portfolio, and the returns of portfolios of medium or high B/M ratios are perceived as more ambiguous. We know no clear reason for these patterns, but we see that the ordering of the FF6 portfolios with respect to the ambiguity perception coincides with the ordering with respect to the Sharpe ratio. In fact, we will show in the next subsection that this is not a coincidence but the rule.

The measures of ambiguity perception in our example should be contrasted with the assumption that Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010) used to explain the value effect. They argued that since firms of growth stocks have more growth potential, it seems natural to assume that their returns are more ambiguous than those of value stocks. They observed, in the experimental markets with heterogenous investors, that more risk-averse investors tend to be more ambiguity-averse as well. They deduced, from this observation, that growth stocks are held and priced primarily by investors who are less averse to both risk and ambiguity, while value stocks are held and priced more evenly by all investors in the market. They, then, concluded that value stocks have higher returns than growth stocks at equilibrium with heterogeneous investors. In both their experiments and the U.S. equity market data at Ken French's website, the value effect is prevalent. Yet, we found, in the analysis of an ambiguity-averse representative investor based on the U.S. equity market data, that the value stocks tend to have more, rather than less, ambiguous returns. As the settings are so different between their and our papers, it is difficult to pin down where this difference is from.

We have so far looked into the ambiguity perceived in the return each of the FF6 portfolios, but the ambiguity perceived in the return of the optimal portfolio is also worthy of attention. It is written, in symbols, as $\mathrm{Amb}\,(x \mid \Sigma_M^{\theta*})$, where x is one of the three portfolios in Table 2, and $\Sigma_M^{\theta*}$ is the second-order belief that attains the implied ambiguity for the respective portfolio. This ambiguity measure tells us how much ambiguity is perceived in the return of the optimal portfolio x relative to other portfolios, especially the FF6 portfolios. For this, using (24) and (25), we can show, with no proof given here, that $\mathrm{Amb}\,\left(x \mid \Sigma_M^{\theta*}\right) = (2+\eta^{\theta*})^{-1/2}$. This equality implies the higher the implied ambiguity, the less ambiguity is perceived in the optimal portfolio relative to other portfolios. This is numerically confirmed in Tables 3 and 4: The implied ambiguity is ranked in the ascending order, but the ambiguity measure of the optimal portfolio is ranked in the descending order, as the equal-weight, value-weight, and GMV portfolios.

6.3 Sensitivity of our quantitative results

So far, we have mainly focused on the coefficient θ^* of absolute risk aversion that minimizes the coefficient η^θ of relative ambiguity aversion. In the following, we examine to what extent our numerical results depend on the choice of the coefficient of absolute risk aversion.

The left figure in Figure 2 is the graph of the coefficients η^θ of relative ambiguity aversion as a function of the coefficients θ of the risk aversion, when the value-weighted portfolio in Table 2 is optimal. We can see that while, by its definition (19), the function η^θ diverges to infinity as θ approaches to its upper bound $\overline{\theta}$ or the lower bound 0, it does not vary much around the coefficient θ^* at which the function attains its minimum.

The right figure in Figure 2 is the graph of the measures of ambiguity perception, defined by (23), of the FF6 portfolios as a function of the coefficients θ of the risk aversion, when the value-weighted portfolio in Table 2 is optimal. The ambiguity measures are decreasing functions of θ, except for the SL portfolio near the upper bound $\bar{\theta}$. While the ranking of ambiguity measures of the FF6 portfolios changes around $\bar{\theta}$, it does not change around the coefficient θ^* at which the coefficient η^θ of relative ambiguity aversion is minimized.

The numerical results of the right figure in Figure 2 are consistent to the following two theoretical findings, which we state here without proof. First, for every $v \in \boldsymbol{R}^N$, as θ approaches 0, the measure of ambiguity approaches $\mathrm{Sr}\,(v)/\mathrm{Sr}\,(\overline{x})$. That is, when the investor is almost risk-neutral, the ranking by the measures of ambiguity perception is equal to the ranking by the Sharpe ratios. This can be easily confirmed along the vertical axis of the right figure in Figure 2. Although the measure of ambiguity perception depends, in general, on the choice of a portfolio x, these limits, as θ approaches zero, are independent of the choice of x. Moreover, this ranking persists except when θ is very close to the upper bound $\bar{\theta}$. That the returns of the SH, SN, and BH portfolios, two of which consist of value stocks, are perceived as more ambiguous than others is, therefore, a fairly robust phenomenon.

The second theoretical finding is that for every $v \in \boldsymbol{R}^N \backslash \{0\}$, as θ approaches the upper bound $\bar{\theta}$,

$$\mathrm{Amb}\,(v \,|\, \Sigma_M^\theta) \to \frac{|\alpha(x)^\top v|}{(v^\top \Sigma_R v)^{1/2}} \frac{1}{\left(\alpha(x)^\top \Sigma_R^{-1} \alpha(x)\right)^{1/2}}.$$

The first fraction of the limit is similar to the Sharpe ratio of the portfolio return, but different from it in that the numerator is the absolute value of the alpha of the portfolio return. The second fraction is the normalizing factor that guarantees that the ambiguity measure $\mathrm{Amb}\,(v \,|\, \Sigma_M^\theta)$ lies between 0 and 1. In the right figure of Figure 2, we see that the return of the SL portfolio is perceived as increasingly more ambiguous than others as θ approaches $\bar{\theta}$. This is due to its negative, but large in absolute value, alpha, which can be found in Table 2.

We also examine to what extent the coefficients of relative ambiguity aversion and the second-order beliefs depend on the specification of the value-weighted portfolio. In the previous subsections, we derived $\overline{\theta}$, θ^*, and η^{θ^*} from the averages of the fractions of market capitalizations of the FF6 portfolios, where the averages are taken over the sample period of July 1926 to December 2017. Here, we derive them from each of the ever observed fractions of the market capitalizations of the FF6 portfolios during the same sample period. We then obtain 1098 samples of $\overline{\theta}$, θ^*, and η^{θ^*} (as there are 1098 months in the sample period), and put their means, standard deviations, minima, and maxima in Table 5. We see that the standard deviations of $\overline{\theta}$, θ^*, and η^{θ^*} are small and neither their minima nor maxima are far from their means. Table 6 reports the means, standard deviations, minima, and maxima of the ambiguity measures of the FF6 portfolios. Again, we see that the standard deviations are small and neither the minima nor maxima are far from the means.

7 Conclusion

In the ambiguity-inclusive CARA-normal setup, we have identified necessary and sufficient conditions for a given portfolio, different from the tangency portfolio, to be optimal for some ambiguity-averse investor; characterized the second-order beliefs and the coefficients of relative ambiguity aversion with which the given portfolio is optimal; and shown that the implied ambiguity, that is, the smallest coefficient of relative ambiguity aversion with which the portfolio is optimal, is a function of the Sharpe ratio of the portfolio return. We have numerically derived the implied ambiguity and the associated second-order belief, along with the ambiguity measures, for the FF6 portfolios of the U.S. equity markets.

There are a couple of directions of future research. First, we should quantify the deviation of various portfolios observed or recommended to hold from the tangency portfolio in terms of the implied ambiguity. There are many types of portfolios that are different from the tangency portfolio and yet observed or recommended to hold. Other than those which we have already dealt with, that is, the value-weighted, the equal-weighted, and GMV portfolios, there are many more examples in Ang (2014, Section 5 of Chapter 3). These portfolios may well be optimal in different contexts, but by quantifying them in terms of the implied ambiguity, we can compare them on a common scale, and such direct comparisons would be of some help to investors' portfolio choice.

Second, we should grasp a better understanding on the estimator properties of the implied ambiguity. Since the means, variances, and covariances of asset returns need to be estimated, the Sharpe ratios and the implied ambiguity need to be estimated as well. Without knowing their distributions, either the small-sample ones or asymptotic ones, we cannot rely much on our numerical results. These distributional properties will be better understood by referring to the vast literature on empirical asset pricing and tests of the CAPM.

Tables and Figures

Table 1: Sample means, variances, and covariances of the FF6 Portfolios from July 1926 to Dec 2017

	R_{f} and μ (%)	Σ_R					
R_{f}	0.28						
SL	0.98	56.40	49.71	54.63	33.86	34.58	43.51
SN	1.26	49.71	48.51	54.52	31.09	34.72	44.56
SH	1.47	54.63	54.52	66.18	33.74	40.03	53.14
BL	0.92	33.86	31.09	33.74	27.91	26.66	31.30
BN	0.97	34.58	34.72	40.03	26.66	31.97	37.62
BH	1.20	43.51	44.56	53.14	31.30	37.62	50.80

Table 1 reports the sample mean of the 1-month TBill return and the sample means and covariances of the monthly returns of the FF6 portfolios, obtained from Ken French's website. In our numerical example, the sample mean of the 1-month TBill return is used as R_{f}, and the sample means and covariances are used as μ and Σ_R. The FF6 portfolios are formed in the following manner. First, sort out the stocks traded on NYSE, AMEX, and NASDAQ in terms of the market capitalizations (also known as the market values and the market equities), and the ratio of the book equity (also known as the book value) to the market equity, abbreviated as B/M. Second, partition the stocks, with positive book equity, into six groups, according to whether the market capitalization belongs to the top 50% or the bottom 50% (referred to as being Big or Small), and whether the B/M belongs to the top 30%, the bottom 30%, or neither (referred to as being High, Low, or Neutral). Third, form the value-weighted portfolio of the stocks in each of the six portfolios. The six portfolios thus formed are named SL, SN, SH, BL, BN, and BH in the obvious manner.

Table 2: CAPM alphas and betas of the FF6 portfolios and the weights in the tangency, value-weighted, equal-weighted, and GMV portfolios.

	Tangency Portfolio	Value-weighted Portfolio	Equal-weighed Portfolio	GMV Portfolio	CAPM alpha	CAPM beta	Sharpe Ratio
SL	-3.10	0.02	0.23	-0.56	-0.15	1.20	0.09
SN	3.07	0.03	0.24	0.80	0.16	1.16	0.14
SH	1.09	0.02	0.28	-0.34	0.27	1.30	0.15
BL	1.96	0.51	0.11	0.87	-0.02	0.93	0.12
BN	-1.29	0.31	0.10	0.73	-0.02	0.99	0.12
BH	-0.73	0.11	0.04	-0.50	0.07	1.20	0.13
ER	2.09	0.99	1.19	0.86			
SD	8.36	5.51	6.82	4.75			
SR	0.22	0.13	0.13	0.12			

Table 2 reports the weights in each of the tangency, value-weighted, equal-weighed portfolio, and GMV (global minimum variance) portfolios. The weights in the tangency portfolio are computed from the sample means, variances, and covariances of the returns of the FF6 portfolios. The value-weighted portfolio is determined by the time-series averages of the weights of the market capitalizations of the FF6 portfolios. The equal-weighed portfolio is determined by the time-series averages of the factions of numbers of the firms in each of the FF6 portfolio. The weights in the GMV portfolio are computed from the sample variances and covariances of the returns of the FF6 portfolios. Each portfolio is normalized so that the coordinates add up to one. The expected returns (denoted as ER), standard deviations (denoted as SD), and Sharpe ratios (denoted as SR) of the four portfolios are also reported. Table 2 also reports the CAPM alpha, beta, and Sharpe ratio of the FF6 portfolios. The CAPM beta is computed using the value-weighted portfolio and the sample variances and covariances. Then CAPM alpha is computed using the beta.

Table 3: The implied ambiguity, the Sharpe ratio, and the norm of pricing error

	$\overline{\theta}$	θ^*	η^{θ^*}	$\dfrac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}$	$\left(\dfrac{\alpha(x)^\top \Sigma_R^{-1}\alpha(x)}{\alpha(\underline{x})^\top \Sigma_R^{-1}\alpha(\underline{x})}\right)^{1/2}$
Value-Weighted	0.0234	0.0130	8.2534	0.5934	0.8049
Equal-Weighted	0.0196	0.0109	7.5018	0.6137	0.7895
GMV	0.0260	0.0143	9.3015	0.5680	0.8230

Table 3 reports the implied ambiguity and the corresponding coefficients of absolute risk aversion of the value-weighted, equal-weighed, and GMV portfolios in Table 2. The upper bound $\overline{\theta}$ on the coefficients of constant absolute risk aversion is defined in (15). The coefficients θ^* of constant absolute risk aversion and the implied ambiguity η^{θ^*} are defined in Theorem 4. $\mathrm{Sr}\,(x)/\,\mathrm{Sr}\,(\overline{x})$ is the Sharpe ratio of each portfolio divided by the Shape ratio of the tangency portfolio $\overline{x}$. $\left(\alpha(x)^\top \Sigma_R^{-1}\alpha(x)/\alpha(\underline{x})^\top \Sigma_R^{-1}\alpha(\underline{x})\right)^{1/2}$ is the norm, defined in (28), of the vector of alphas if we use each portfolio as the pricing factor, divided by the norm of the return of a portfolio $\underline{x}$ with a zero expected excess return.

Table 4: Measure of ambiguity perception

	SL	SN	SH	BL	BN	BH	Potfolio
Value-Weighted	0.17	0.41	0.45	0.28	0.29	0.34	0.31
Equal-Weighted	0.13	0.37	0.41	0.31	0.30	0.34	0.32
GMV	0.28	0.51	0.57	0.32	0.35	0.45	0.30

Table 4 reports the measure of ambiguity perception, associated with the smallest coefficient of relative ambiguity aversion, which is defined in (9), of the FF6 portfolios when the value-weighted, equal-weighted, and GMV portfolios in Table 2 are optimal. The measure of ambiguity perception with the second-order belief $\Sigma_M^{\theta*}$ is given in (23).

Table 5: Sensitivity of the implied ambiguity and the corresponding coefficient of absolute risk aversion to the weights in the value-weighted portfolio

	$\overline{\theta}$	θ^*	η^{θ^*}
mean	0.0234	0.0129	8.2931
s.d.	0.0003	0.0002	0.1832
min	0.0222	0.0123	7.9377
max	0.0238	0.0131	8.9771

For each ever observed weights in the value-weighted portfolio during the sample period, we compute parameters $\overline{\theta}$, θ^*, and η^{θ^*}. Table 5 reports the means, standard deviations, minima, and maxima of these computed values.

Table 6: Sensitivity of the measure of ambiguity perception of FF6 portfolios to the weights in the value-weighted portfolio

	SL	SN	SH	BL	BN	BH
mean	0.17	0.41	0.45	0.29	0.29	0.34
s.d.	0.00	0.01	0.01	0.00	0.01	0.01
min	0.16	0.40	0.43	0.28	0.28	0.33
max	0.18	0.43	0.48	0.29	0.31	0.37

For each ever observed weights in the value-weighted portfolio during the sample period, we compute the ambiguity measure of FF6 the portfolios. Table 6 reports the means, standard deviations, minima, and maxima of these computed values.

Figure 1: Mean-variance efficiency frontier and the tangency portfolio

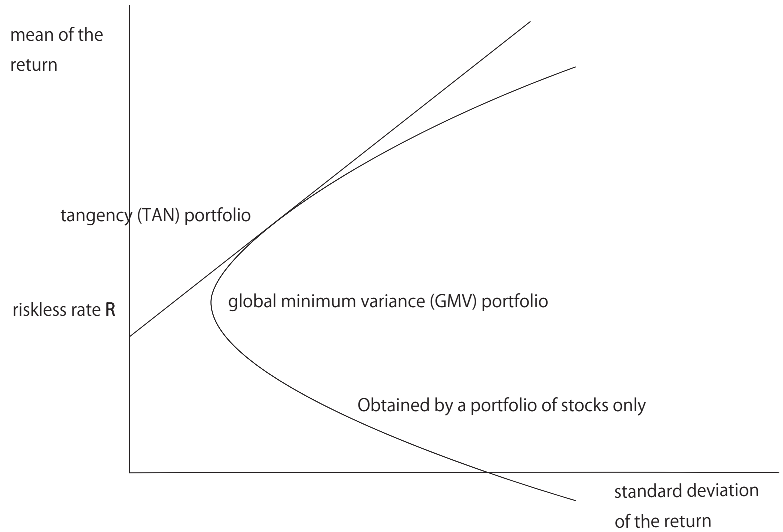
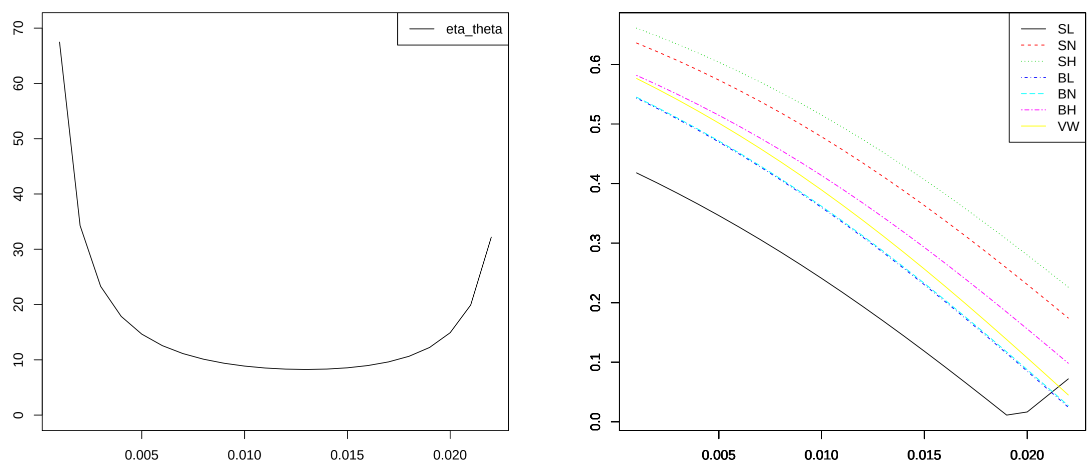


Figure 1 depicts the mean-variance efficiency frontier and the riskless return in the space of means and standard deviations, and the highest Sharpe ratio, attained by the tangency portfolio.

Figure 2: Coefficients η^θ of relative ambiguity aversion as a function of the coefficients θ of absolute risk aversion, and ambiguity measure of the FF6 and market portfolios



The left figure is the graph of the coefficients η^θ, which is defined in (19), of relative ambiguity aversion as a function of the coefficients θ of absolute risk aversion when the value-weighted portfolio in Table 2 is optimal. The right figure gathers the graphs of the ambiguity measures $\mathrm{Amb}\,(\cdot \mid \Sigma_M^\theta)$ of the FF6 and market portfolios as functions of the coefficients of absolute risk aversion when the value-weighted portfolio in Table 2 is optimal. The ambiguity measure is given by equation (23).

A Lemmas and Proofs for Section 2

Proof of Lemma 1 By the properties of the moment generating function,

$$\begin{aligned}
E\left[u_\theta\left(x^\top R + yR_\mathrm{f}\right)|M\right] &= -\exp(-\theta y R_\mathrm{f}) E\left[\exp\left((-\theta x)^\top R\right)|M\right] \\
&= -\exp(-\theta y R_\mathrm{f}) \exp\left((-\theta x)^\top M + \frac{1}{2}(\theta x)^\top \Sigma_{R|M}(\theta x)\right) \\
&= -\exp\left(-\theta\left(x^\top M + R_\mathrm{f} y - \frac{\theta}{2} x^\top \Sigma_{R|M} x\right)\right).
\end{aligned}$$

Then it follows from the definition of $\varphi_{\gamma,\theta}$ that

$$\varphi_{\gamma,\theta}\left(E\left[u_\theta\left(x^\top R + yR_\mathrm{f}\right)|M\right]\right) = -\exp\left(-\gamma\left(x^\top M + R_\mathrm{f} y - \frac{\theta}{2} x^\top \Sigma_{R|M} x\right)\right).$$

Thus, again by the properties of the moment generating function,

$$\begin{aligned}
U_{\gamma,\theta}(x^\top R + yR_\mathrm{f}) &= U_{\gamma,\theta}(x^\top R + yR_\mathrm{f}) E\left[-\exp\left(-\gamma\left(x^\top M + R_\mathrm{f} y - \frac{\theta}{2} x^\top \Sigma_{R|M} x\right)\right)\right] \\
&= -\exp\left(-\gamma\left(R_\mathrm{f} y - \frac{\theta}{2} x^\top \Sigma_{R|M} x\right)\right) E\left[\exp\left((-\gamma x)^\top M\right)\right] \\
&= -\exp\left(-\gamma\left(R_\mathrm{f} y - \frac{\theta}{2} x^\top \Sigma_{R|M} x\right)\right) \exp\left(\mu^\top(-\gamma x) + \frac{1}{2}(-\gamma x)^\top \Sigma_M (-\gamma x)\right) \\
&= u_\gamma\left(V_{\gamma,\theta}(x,y)\right).
\end{aligned}$$

///

To prove Proposition 2, we need a lemma. To state it, let $P \in \mathscr{S}^N$ and consider the problem of minimizing the quadratic form defined by P:

$$\min_{w \in \boldsymbol{R}^N \setminus \{0\}} \frac{w^\top P w}{\|w\|^2}. \tag{30}$$

Denote a solution by w_1. Next, let $n \geqslant 2$ and $w_1, w_2, \dots w_{n-1}$ belong to $\boldsymbol{R}^N$, and consider the problem of minimizing the quadratic form subject to the constraint that the solution must be orthogonal to $w_1, w_2, \dots, w_{n-1}$:

$$\begin{aligned}
&\min_{w \in \boldsymbol{R}^N \setminus \{0\}} && \frac{w^\top P w}{\|w\|^2} \\
&\text{s. t.} && w_m \cdot w = 0 \text{ for every } m \leqslant n-1.
\end{aligned} \tag{31}$$

We say that a sequence $(w_1, w_2, \ldots, w_N)$ of vectors in $\boldsymbol{R}^N$ is a sequence of solutions to (30) and (31) if $w_1, w_2, \ldots, w_N$ are obtained iteratively by solving (30) and (31). There is a sequence of solutions to the problems (31), because the objective functions are continuous and the domains can be restricted to $\{w \in \boldsymbol{R}^N \mid \|w\| = 1\}$. Moreover, for every sequence $(w_1, w_2, \ldots, w_N)$ of solutions, $(w_1, w_2, \ldots, w_N)$ is orthogonal and

$$\frac{w_1^\top P w_1}{\|w_1\|^2} \leqslant \frac{w_2^\top P w_2}{\|w_2\|^2} \leqslant \cdots \leqslant \frac{w_N^\top P w_N}{\|w_N\|^2}$$

for every n. Since the following lemma, which characterizes the eigenvectors and eigenvalues of P, is well known,[15] we omit the proof.

Lemma 3 *1. For every sequence $(w_1, w_2, \ldots, w_N)$ of solutions to the sequence of problems* (31) *and for every n, w_n is an eigenvector of P and its corresponding eigenvalue is equal to $w_n^\top P w_n / \|w_n\|^2$.*

2. For every sequence $(w_1, w_2, \ldots, w_N)$ of eigenvectors of P that is orthogonal and of which the sequence of the corresponding eigenvalues is non-decreasing is a sequence of solutions to the sequence of problems (31)*.*

3. If $w_1, w_2, \ldots, w_K$ are eigenvectors of P that correspond to distinct eigenvectors, then $(w_1, w_2, \ldots, w_K)$ is orthogonal.

Proposition 2 can be proved using this lemma as follows.

Proof of Proposition 2 Define $P = \Sigma^{-1/2}\Xi\Sigma^{-1/2}$. Since $\Sigma_M \in \mathscr{S}^N$, $P \in \mathscr{S}^N$. Moreover, assume that $w \in \boldsymbol{R}^N$, $v \in \boldsymbol{R}^N$, $\lambda \in \boldsymbol{R}$ and $v = \Sigma^{-1/2} w$. Then $(v^\top \Xi v)/(v^\top \Sigma v) = (w^\top P w)/\|w\|^2$, and $(\Sigma^{-1}\Xi)v = \lambda v$ if and only if $Pw = \lambda w$. Thus, parts 1 and 2 of this proposition follow from parts 1 and 2 of Lemma 3. If, in addition, $w' \in \boldsymbol{R}^N$, $v' \in \boldsymbol{R}^N$, and $v' = \Sigma^{-1/2} w'$, then $v^\top \Sigma v = w \cdot w'$. Thus, part 3 of this proposition follows from part 3 of Lemma 3. It remains to prove part 4. Let λ be an eigenvalue of Q and v be a corresponding eigenvector. Then $\Sigma_M v = \lambda \Sigma_R v$. Thus $v^\top \Sigma_M v = \lambda v_n^\top \Sigma_R v$, that is,

$$\lambda = \frac{v^\top \Sigma_M v}{v^\top \Sigma_M v + v^\top \Sigma_{R|M} v}.$$

[15] It is used, for example, in Campbell, Lo, and MacKinlay (1997, Section 6.4), in which the minimum is replaced by the maximum.

Since $v^\top \Sigma_{R|M} v \geqslant 0$, $0 \leqslant \lambda \leqslant 1$. ///

B Proofs for Section 3

Proof of Theorem 1 Let Λ be the set of all eigenvalues of Q. It follows from Proposition 2 that $\Lambda \subset [0,1]$. For each $\lambda \in \Lambda$, denote by V_λ the eigenspace that correspond to λ. It also follows that V_λ is a linear subspace of $\boldsymbol{R}^N$, $(V_\lambda)_{\lambda\in\Lambda}$ is linearly independent (that is, if $v_\lambda \in V_\lambda$ for every $\lambda \in \Lambda$ and $\sum_{\lambda\in\Lambda} v_\lambda = 0$, then $v_\lambda = 0$ for every $\lambda \in \Lambda$), and $\sum_{\lambda\in\Lambda} V_\lambda = \boldsymbol{R}^N$.

Then, for each $\lambda \in \Lambda$, there is a $v_\lambda \in V_\lambda$ such that $\zeta(0) = \sum_{\lambda\in\Lambda} v_\lambda$. Since $\zeta(0) \neq 0$, there is a $\lambda \in \Lambda$ such that $v_\lambda \neq 0$. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_K\}$ be the set of all such λ's. For each k, write $v_k = v_{\lambda_k}$, then $\zeta(0) = \sum_{k=1}^K v_k$.

Since $(I + \eta Q)v_k = (1 + \eta\lambda_k)v_k$, $(I + \eta Q)^{-1}v_k = (1 + \eta\lambda_k)^{-1}v_k$. Thus,

$$\zeta(\eta) = (I + \eta Q)^{-1}\zeta(0) = \sum_{k=1}^K (I + \eta Q)^{-1} v_k = \sum_{k=1}^K \frac{1}{1 + \lambda_k \eta} v_k.$$

///

Proof of Proposition 1 We first prove the following fact: let K be a positive integer and suppose that there are K distinct nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_K$ such that $\operatorname{rank}(\lambda_k\Sigma_R - \Sigma_M) < N$ for every k and $\sum_{k=1}^K \operatorname{rank}(\lambda_k\Sigma_R - \Sigma_M) = (K-1)N$. Then, for every k, there exists a $v_k \in \operatorname{Ker}(\lambda_k\Sigma_R - \Sigma_M)$ such that, for every $\eta > -1$, $\zeta(\eta) = \sum_{k=1}^K (1+\lambda_k\eta)^{-1} v_k$. Indeed, by the definition of Q, $\operatorname{Ker}(\lambda_k\Sigma_R - \Sigma_M) = \operatorname{Ker}(\lambda_k I_N - Q)$ and $\dim \operatorname{Ker}(\lambda_k I_N - Q) = N - \operatorname{rank}(\lambda_k\Sigma_R - \Sigma_M) > 0$. Thus $\operatorname{Ker}(\lambda_k\Sigma_R - \Sigma_M)$ is the eigenspace of Q that corresponds to eigenvalue λ_k. Moreover,

$$\sum_{k=1}^K \dim \operatorname{Ker}(\lambda_k\Sigma_R - \Sigma_M) = \sum_{k=1}^K (N - \operatorname{rank}(\lambda_k\Sigma_R - \Sigma_M)) = KN - (K-1)N = N.$$

Thus $\boldsymbol{R}^N$ coincides with the direct sum of the K eigenspaces $\operatorname{Ker}(\lambda_k\Sigma_R - \Sigma_M)$. Thus $\lambda_1, \lambda_2, \ldots, \lambda_K$ are the eigenvalues of Q. Part 1 therefore follows from Theorem 1.

1. This is a special case of the above fact with $K = 1$.

2. Since $\operatorname{rank}\Sigma_M < N$ and $\operatorname{rank}(\Sigma_R - \Sigma_M) < N$, this is a special case of the above fact with $K = 2$ and $\{\lambda_1, \lambda_2\} = \{0, \lambda\}$.

3. Since $\operatorname{rank} Q = \operatorname{rank} \Sigma_M = 1$, by Proposition 2, there are a $\lambda > 0$ and $v \in \boldsymbol{R}^N \backslash \{0\}$ such that $Qv = \lambda v$, that is, $(\lambda \Sigma_R - \Sigma_M) v = 0$. Thus $\operatorname{rank}(\lambda \Sigma_R - \Sigma_M) \leqslant N$. Hence $\operatorname{rank}(\lambda \Sigma_R - \Sigma_M) + \operatorname{rank} \Sigma_M \leqslant N$. On the other hand, $\operatorname{rank}(\lambda \Sigma_R - \Sigma_M) + \operatorname{rank} \Sigma_M \geqslant \operatorname{rank} \lambda \Sigma_R = N$. Thus $\operatorname{rank}(\lambda \Sigma_R - \Sigma_M) + \operatorname{rank} \Sigma_M = N$. The conclusion follows then from part 2. ///

C Lemmas and Proofs for Section 4

Proof of Theorem 2 Note first that (7) can be more succinctly written as $(\eta \Sigma_M) x = \Sigma_R v^\theta$. Since $v^\theta \neq 0$,

$$0 < x^\top (\eta \Sigma_M) x = x^\top \Sigma_R v^\theta = \frac{1}{\theta} (\mu - R_\mathrm{f} \mathbf{1}) \cdot x - x^\top \Sigma_R x.$$

(14) and (16) follow from this. By the Cauchy-Schwartz inequality,

$$v^\top (\eta \Sigma_M) v \geqslant \frac{\left(v^\top (\eta \Sigma_M) x\right)^2}{x^\top (\eta \Sigma_M) x} = \frac{\left(v^\top \Sigma_R v^\theta\right)^2}{x^\top \Sigma_R v^\theta} = v^\top \left(\frac{1}{x^\top \Sigma_R v^\theta} \left(\Sigma_R v^\theta\right) \left(\Sigma_R v^\theta\right)^\top \right) v$$

for every $v \in \boldsymbol{R}^N$. This is equivalent to (20). Since $\Sigma_R \geqslant \Sigma_M$, $\Sigma_R \geqslant \dfrac{1}{\eta} \dfrac{1}{x^\top \Sigma_R v^\theta} \left(\Sigma_R v^\theta\right) \left(\Sigma_R v^\theta\right)^\top$, that is, $\eta \geqslant \left(v^\top \left(\dfrac{1}{x^\top \Sigma_R v^\theta} \left(\Sigma_R v^\theta\right) \left(\Sigma_R v^\theta\right)^\top \right) v \right)^{-1} (v^\top \Sigma_R v)$ for every $v \in \boldsymbol{R}^N$. By Proposition 2, the maximum of the right-hand side attained by varying v is equal to the largest eigenvalue of the matrix

$$\Sigma_R^{-1} \left(\frac{1}{x^\top \Sigma_R v^\theta} \left(\Sigma_R v^\theta\right) \left(\Sigma_R v^\theta\right)^\top \right) = \frac{1}{x^\top \Sigma_R v^\theta} v^\theta \left(\Sigma_R v^\theta\right)^\top$$

with a corresponding eigenvector v^θ. (21) thus follows. ///

Proof of Theorem 3 It follows from its definition that $\Sigma_M^\theta \in \mathscr{S}_+^N$. Since $(\mu - R_\mathrm{f} \mathbf{1}) \cdot x > 0$, $x^\top \Sigma_R v^\theta > 0$ and, thus, $\eta^\theta > 0$. (22) holds if and only if $\eta^\theta \Sigma_M^\theta x = \Sigma_R v^\theta$, and the latter equality follows from definition. It remains to prove that $\Sigma_R \geqslant \Sigma_M^\theta$. By definition and the Cauchy-Schwartz inequality,

$$v^\top \Sigma_M^\theta v = \frac{\left(v^\top \Sigma_R v^\theta\right)^2}{(v^\theta)^\top \Sigma_R v^\theta} \leqslant \frac{\left(v^\top \Sigma_R v\right) \left((v^\theta)^\top \Sigma_R v^\theta\right)}{(v^\theta)^\top \Sigma_R v^\theta} = v^\top \Sigma_R v$$

for every $v \in \boldsymbol{R}^N$. Thus $\Sigma_R \geqslant \Sigma_M^\theta$. ///

Theorem 4 is proved via the following Lemma 4.

Lemma 4 *Let $x \in \boldsymbol{R}^N$, $c \in \boldsymbol{R}^N$, $\Sigma \in \mathscr{S}_{++}^N$, and $S \in \mathscr{S}_{++}^N$. For each $\theta > 0$, define*

$$v^\theta = \frac{1}{\theta}\Sigma^{-1}c - x$$

and assume that there is no $\theta > 0$ for which $v^\theta = 0$. Assume also that $x \cdot c > 0$ and write $\bar{\theta} = \left(x^\top \Sigma x\right)^{-1}(x \cdot c)$. Then there is a unique $\theta > 0$ that minimizes the function

$$\begin{array}{ccc} \left(0, \bar{\theta}\right) & \rightarrow & \boldsymbol{R}_{++} \\ \theta & \mapsto & \dfrac{\left(v^\theta\right)^\top S v^\theta}{x^\top \Sigma v^\theta}, \end{array} \tag{32}$$

which is given by

$$\frac{1}{\theta} = \frac{1}{\bar{\theta}} + \left(\frac{\left(v^{\bar{\theta}}\right)^\top S v^{\bar{\theta}}}{c^\top \Sigma^{-1} S \Sigma^{-1} c}\right)^{1/2} \tag{33}$$

and the minimized value of the function is equal to

$$\frac{2}{x \cdot c}\left(\left(c^\top \Sigma^{-1} S \Sigma^{-1} c\right)^{1/2}\left(\left(v^{\bar{\theta}}\right)^\top S v^{\bar{\theta}}\right)^{1/2} + \left(v^{\bar{\theta}}\right)^\top S \Sigma^{-1} c\right). \tag{34}$$

To find the minimum of η^θ over $\theta \in (0, \bar{\theta})$, we take $S = \Sigma$. Then (33) can be rewritten as

$$\frac{1}{\theta} = \frac{1}{\bar{\theta}} + \left(\frac{\left(\bar{\theta}^{-1}c - \Sigma x\right)^\top \Sigma^{-1}\left(\bar{\theta}^{-1}c - \Sigma x\right)}{c^\top \Sigma^{-1} c}\right)^{1/2},$$

and (34) can be rewritten as

$$\frac{2}{x \cdot c}\left(\left(\left(c^\top \Sigma^{-1} c\right)\left(\bar{\theta}^{-1}c - \Sigma x\right)^\top \Sigma^{-1}\left(\bar{\theta}^{-1}c - \Sigma x\right)\right)^{1/2} + \left(\bar{\theta}^{-1}c - \Sigma x\right)^\top \Sigma^{-1} c\right).$$

To find the minimum, over $\theta \in (0, \bar{\theta})$, of the largest eigenvalue of $\left(x^\top \Sigma v^\theta\right)^{-1}\left(\Sigma v^\theta\right)\left(\Sigma v^\theta\right)^\top \in \mathscr{S}_+^N$, we take $S = \Sigma^2$. Then (33) can be rewritten as $1/\theta = 1/\bar{\theta} + \left\|\bar{\theta}^{-1}c - \Sigma x\right\|/\|c\|$, and (34) can be rewritten as $(2/(x \cdot c))\left(\|c\|\|\bar{\theta}^{-1}c - \Sigma x\| + \left(\bar{\theta}^{-1}c - \Sigma x\right) \cdot c\right)$.

Proof of Lemma 4 The function (32) is continuous. Its value diverges to infinity as

$\theta \uparrow \bar{\theta}$ (because, then, $x^\top \Sigma v^\theta \to 0$) and as $\theta \downarrow 0$. Thus there is a θ that minimizes the function. To show that (33) gives the unique solution, write $\beta = \theta^{-1}$ and differentiate

$$\ln \frac{\left(\beta\Sigma^{-1}c - x\right)^\top S \left(\beta\Sigma^{-1}c - x\right)}{x^\top \Sigma \left(\beta\Sigma^{-1}c - x\right)}$$

with respect to β to obtain the first-order necessary condition

$$\frac{2(\beta\Sigma^{-1}c - x)^\top S\Sigma^{-1}c}{\left(\beta\Sigma^{-1}c - x\right)^\top S \left(\beta\Sigma^{-1}c - x\right)} - \frac{x \cdot c}{\beta x \cdot c - x^\top \Sigma x} = 0,$$

which is equivalent to

$$\left(\beta - \frac{1}{\bar{\theta}}\right)^2 - \frac{\left(v^{\bar{\theta}}\right)^\top S v^{\bar{\theta}}}{c^\top \Sigma^{-1} S \Sigma^{-1} c} = 0. \tag{35}$$

Since $v^{\bar{\theta}} \neq 0$, the second term of the left-hand side is strictly positive. Hence (35) has two distinct solutions, one larger and the other smaller than $\bar{\theta}^{-1}$. The former satisfies the first-order condition but the latter does not, because it is necessary that $\beta > \bar{\theta}^{-1}$. Therefore, there is only one solution to the problem of minimizing the function (32), which is given by (33).

As for the minimized value of the function, note that $v^\theta = v^{\bar{\theta}} + \left(\frac{\left(v^{\bar{\theta}}\right)^\top S v^{\bar{\theta}}}{c^\top \Sigma^{-1} S \Sigma^{-1} c}\right)^{1/2} \Sigma^{-1} c$. Thus $x^\top \Sigma v^\theta = (x \cdot c) \left(\left(v^{\bar{\theta}}\right)^\top S v^{\bar{\theta}} / c^\top \Sigma^{-1} S \Sigma^{-1} c\right)^{1/2}$ and

$$\begin{aligned}
\left(v^\theta\right)^\top S v^\theta &= \left(v^{\bar{\theta}}\right)^\top S v^{\bar{\theta}} + 2\left(\frac{\left(v^{\bar{\theta}}\right)^\top S v^{\bar{\theta}}}{c^\top \Sigma^{-1} S \Sigma^{-1} c}\right)^{1/2} \left(v^{\bar{\theta}}\right)^\top S \Sigma^{-1} c + \frac{\left(v^{\bar{\theta}}\right)^\top S v^{\bar{\theta}}}{c^\top \Sigma^{-1} S \Sigma^{-1} c} c^\top \Sigma^{-1} S \Sigma^{-1} c \\
&= 2\left(v^\theta\right)^\top S v^\theta + 2\left(\frac{\left(v^\theta\right)^\top S v^\theta}{c^\top \Sigma^{-1} S \Sigma^{-1} c}\right)^{1/2} v^{\bar{\theta}} S \Sigma^{-1} c.
\end{aligned}$$

Thus (34) is obtained. ///

We can now prove Theorem 4.

Proof of Theorem 4 By applying Lemma 4 to the case where $c = \mu - R_\mathrm{f}\mathbf{1}$, $\Sigma = \Sigma_R$, and $S = \Sigma$, as explained right after the lemma, we see that η^θ is maximized when θ

is equal to

$$\left(\frac{1}{\bar{\theta}}+\left(\frac{\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)^{\top}\Sigma_R^{-1}\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)}{(\mu-R_{\mathrm{f}}\mathbf{1})^{\top}\Sigma_R^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})}\right)^{1/2}\right)^{-1} \tag{36}$$

and, then, η^{θ} is equal to

$$\begin{aligned}&\frac{2}{x\cdot(\mu-R_{\mathrm{f}}\mathbf{1})}\times\Bigg[\left(\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)^{\top}\Sigma_R^{-1}\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)\right)^{1/2}\\&\left((\mu-R_{\mathrm{f}}\mathbf{1})^{\top}\Sigma_R^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})\right)^{1/2}+\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)^{\top}\Sigma_R^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})\Bigg].\end{aligned} \tag{37}$$

Note here that $\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)^{\top}\Sigma_R^{-1}\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)$ is equal to

$$\begin{aligned}&\bar{\theta}^{-2}\left(\mathrm{Sr}\ (\overline{x})\right)^2-2\bar{\theta}^{-1}x\cdot(\mu-R_{\mathrm{f}}\mathbf{1})+x^{\top}\Sigma_R x\\=&\bar{\theta}^{-2}\left(\mathrm{Sr}\ (\overline{x})\right)^2-x^{\top}\Sigma_R x=\bar{\theta}^{-2}\left((\mathrm{Sr}\ (\overline{x}))^2-(\mathrm{Sr}\ (x))^2\right).\end{aligned}$$

Thus,

$$\frac{\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)^{\top}\Sigma_R^{-1}\left(\bar{\theta}^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})-\Sigma_R x\right)}{(\mu-R_{\mathrm{f}}\mathbf{1})^{\top}\Sigma_R^{-1}(\mu-R_{\mathrm{f}}\mathbf{1})}=\frac{1}{\bar{\theta}^2}\left(1-\left(\frac{\mathrm{Sr}\ (x)}{\mathrm{Sr}\ (\overline{x})}\right)^2\right).$$

Thus, (36) is equal to

$$\left(\frac{1}{\bar{\theta}}+\frac{1}{\bar{\theta}}\left(1-\left(\frac{\mathrm{Sr}\ (x)}{\mathrm{Sr}\ (\overline{x})}\right)^2\right)^{1/2}\right)^{-1}=\frac{\bar{\theta}}{1+\left(1-\left(\frac{\mathrm{Sr}\ (x)}{\mathrm{Sr}\ (\overline{x})}\right)^2\right)^{1/2}},$$

and (37) is equal to

$$\begin{aligned}&\frac{2}{x\cdot(\mu-R_{\mathrm{f}}\mathbf{1})}\left(\bar{\theta}^{-1}\left((\mathrm{Sr}\ (\overline{x}))^2-(\mathrm{Sr}\ (x))^2\right)^{1/2}\mathrm{Sr}\ (\overline{x})+\bar{\theta}^{-1}\left(\mathrm{Sr}\ (\overline{x})\right)^2-x\cdot(\mu-R_{\mathrm{f}}\mathbf{1})\right)\\=&2\left(\frac{1}{(\mathrm{Sr}\,(x))^2}\left((\mathrm{Sr}\ (\overline{x}))^2-(\mathrm{Sr}\ (x))^2\right)^{1/2}\mathrm{Sr}\ (\overline{x})+\frac{1}{(\mathrm{Sr}\,(x))^2}\left(\mathrm{Sr}\ (\overline{x})\right)^2-1\right)\\=&2\left(\frac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\left(1-\left(\frac{\mathrm{Sr}\,(\overline{x})}{\mathrm{Sr}\,(x)}\right)^2\right)^{1/2}\left(1+\left(1-\left(\frac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2}\right).\end{aligned}$$

Since

$$\left(1+\left(1-\left(\frac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2}\right)\left(1-\left(1-\left(\frac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2\right)^{1/2}\right)=\left(\frac{\mathrm{Sr}\,(x)}{\mathrm{Sr}\,(\overline{x})}\right)^2,$$

this is equal to the right hand side of (25). ///

References

[1] Ahn, D., Choi, S., Gale, D. and Kariv, S. 2014, Estimating ambiguity aversion in a portfolio choice experiment. *Quantitative Economics* 5:195-223. doi: 10.3982/QE243

[2] Ang, A., 2014, *Portfolio Management: A Systematic Approach to Factor Investing*, Oxford University Press.

[3] Attanasi, G., Gollier, C., Montesano, A., and Pace, N. 2014. Eliciting ambiguity aversion in unknown and in compound lotteries: A smooth ambiguity model experimental study. *Theory and decision* 77(4): 485-530.

[4] Bossaerts, P., P. Ghirardato, S. Guarnaschelli, and W.R. Zame, 2010, Ambiguity in asset markets: Theory and experiment, *Review of Financial Studies* 23: 1325–1359.

[5] Canner, N., N. G. Mankiw, and D. N. Weil, 1997, An asset allocation puzzle, *American Economic Review* 87: 181–191.

[6] Campbell, J. Y., A. Lo, and C. MacKinlay. 1997, *The Econometrics of Financial Markets*. Princeton University Press.

[7] Collard, F., S. Mukerji, K. Sheppard, and J. Tallon, 2018, Ambiguity and the historical equity premium, *Quantitative Economics* 945–993.

[8] Duffie, D., 2001, *Dynamic Asset Pricing Theory*, third edition, Princeton University Press.

[9] Ellsberg, D., 1961, Risk, ambiguity, and the Savage axiom, *Quarterly Journal of Economics* 75, 643–669.

[10] Epstein, L. G. 2010, A paradox for the "smooth ambiguity" model of preference, Econometrica 78: 2085–2099.

[11] Fama, E. F., 1996, Multifactor portfolio efficiency and multifactor asset pricing, *Journal of Financial and Quantitative Analysis* 31, 441–465.

[12] Fama, Eugene F., and Kenneth R. French. 1992. The cross-section of expected stock returns. it Journal of finance 47(2):427-465.

[13] Ghirardato, P., F. Maccheroni, and M. Marinacci, 2004, Differentiating ambiguity and ambiguity aversion, *Journal of Economic Theory* 118, 133–173.

[14] Gibbons, M. R., S. A. Ross, and J. Shanken, 1989, A test of the efficiency of a given portfolio, *Econometrica* 57, 1121–1152.

[15] Gilboa, I., and D. Schmeidler. 1989. Maxmin Expected Utility with Non-Unique Prior, *Journal of Mathematical Economics* 18:141–153.

[16] Hansen, L. P., and R. Jagannathan, 1997, Assessing specification errors in stochastic discount factor models, *Journal of Finance* 52, 557–590.

[17] Klibanoff, P., M. Marinacci, and S. Mukerji. 2005. A smooth model of decision making under ambiguity. *Econometrica* 73:1849–1892.

[18] Maccheroni, F., M. Marinacci, and D. Ruffino. 2013. Alpha as Ambiguity: Robust Mean-Variance Portfolio Analysis. *Econometrica* 81:1075–1113.

[19] Harry Markowitz, 1952, Portfolio selection, *Journal of Finance* 7, 77–91.

[20] Rajnish Mehra and Edward C. Prescott, 1985, The equity premium: A puzzle, Journal of Monetary Economics 15, 145–161.

[21] Shanken, J., 1987, Multivariate proxies and asset pricing relations: Living with the Roll critique, *Journal of Financial Economics* 18, 91–110.

[22] Thimme, J., and C. Völkert, 2015, Ambiguity in the cross-section of expected returns: An empirical assessment, *Journal of Business and Economic Statistics* 33, 418–429.

[23] Tobin, J., 1958, Liquidity preference as behavior towards risk, *Review of Economic Studies* 25, 65–86.