Boundary Behavior of Excess Demand Functions without the Strong Monotonicity Assumption

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Abstract

We give a theorem on the existence of an equilibrium price vector for an excess demand correspondence, which may have been derived from consumers with non-monotone preference relations.

1 Introduction

We give a theorem on the existence of an equilibrium price vector for an excess demand correspondence, which may have been derived from consumers with non-monotone preference relations. Several points are noteworthy on this theorem. First, the notion of an equilibrium price vector is an exact one, so that the demand is no more or no less than supply, and hence prices may be negative. Second, the set of prices on which the excess demands are well defined are convex and relatively open. This is a highly non-trivial assumption. Third, only a rather weak condition is imposed on the boundary behavior of the excess demand correspondence.

2 Definitions and Result

Let L be a positive integer and $w \in \boldsymbol{R}^L$. Let $\|\cdot\|$ be the Euclidean norm on $\boldsymbol{R}^L$ and assume that $\|w\| = 1$. Define $H = \{p \in \boldsymbol{R}^L \mid p \cdot w = 1\}$. The interpretation is that L is the number of commodities; w is the direction along which the utility levels of some group of consumers of strictly positive measure can be strictly increased; and H is the set of normalized price vectors.

Theorem 1 *Let $P \subseteq H$ and $\zeta : P \to \boldsymbol{R}^L$ be a correspondence satisfying the following properties.*

1. *P is a nonempty, convex, and open subset of H containing w.*
2. *ζ is nonempty-, convex-, and compact-valued and upper hemi-continuous.*
3. *ζ satisfies the strict Walras law, that is, $p \cdot z = 0$ for every $p \in P$ and $z \in \zeta(p)$.*

4. There exists a nonempty and compact subset V of P such that the set

$$\left\{p \in P \mid \min_{z \in \zeta(p)} \max_{v \in V} v \cdot z < 0\right\} \tag{1}$$

is included in a compact subset of P.

Then there exists a $p^* \in P$ such that $0 \in \zeta(p^*)$.

The first condition may appear to be standard. Note however that P may not be bounded, and that the convexity and openness are non-trivial conditions on the domain of the aggregate excess demand correspondence. The second condition needs no comment. In the third condition, the Walras law is in the strict form, so that we require the strict equality $p \cdot z = 0$ rather than the weak inequality $p \cdot z \leq 0$. The maximum in (1) of the last condition is indeed attained because V is compact. The minimum is also attained because the function $z \mapsto \max_{v \in V} v \cdot z$ is continuous and $\zeta(p_n)$ is compact. Note that negative prices are allowed throughout this set of conditions.

Proof of Theorem 1. The proof method is more or less the same as in Hüsseinov (1999). We assume throughout this section that P, ζ, and V satisfy the conditions of Theorem 1.

Let P^* be a compact subset of P that includes $\left\{p \in P \mid \min_{z \in \zeta(p)} \max_{v \in V} v \cdot z < 0\right\}$. Since P is relatively open in H and $P^* \cup V \cup \{w\}$ is a compact subset of P, there exists a convex and compact subset Q of P such that the relative interior of Q with respect to H includes $P^* \cup V \cup \{w\}$.

By Proposition 3 of Hildenbrand (1974, I.B), there exists a convex and compact subset Z of $\boldsymbol{R}^L$ such that $\zeta(p) \subset Z$ for every $p \in Q$. Following the construction of Debreu (1956), we can show that there exists a $(p^*, z^*) \in Q \times Z$ such that $z^* \in \zeta(p^*)$ and

$$p^* \cdot z^* \geq p \cdot z^* \tag{2}$$

for every $p \in Q$.

Suppose that there is no $c \in \boldsymbol{R}$ such that $z^* = cw$. Then the set $\operatorname{argmax}_{p \in Q} p \cdot z^*$ is included in the relative boundary of Q with respect to H. In particular, p^* belongs to the relative boundary of Q with respect to H. Also, $p \cdot z^* < p^* \cdot z^* = 0$ for every p in the relative interior of Q with respect to H. Since V is included in the relative interior of Q with respect to H, this implies that $\max_{v \in V} v \cdot z^* < 0$ and hence $p^* \in P^*$. But since P^* is also in the relative interior of Q with respect to H, this implies that p^* is in the relative interior of Q with respect to H, which is a contradiction. Hence there exists a $c \in \boldsymbol{R}$ such that $z^* = cw$. By the strict Walras Law and $p^* \cdot w = 1$, if we take the inner products of both sides with p^*, then we obtain $0 = c$. Hence $z^* = 0$ and $0 \in \zeta(p^*)$. ///

Remark 1 While it is desirable to dispense with the assumption that P is relatively open and convex, it is impossible to do so entirely. There is an economy with a continuum of consumers such that P is not relatively open, and in fact there is no equilibrium. For example, the aggregate (mean) excess demand function of Example 10 of Hara (2004) is well defined under a price vector $p \in \boldsymbol{R}^2$ if and only if $p^1 > 0$ and $p^2 \geq 0$. In fact, then, it is a (single-valued) function given by

$$\zeta(p) = \left(\frac{p^2}{p^1}, -1 \right).$$

Any candidate w for the application of the theorem must satisfy $w^1 > 0$ and $w^2 \geq 0$. But then $\{p \in \boldsymbol{R}^2 \mid p^1 > 0, p^2 \geq 0, \text{ and } p \cdot w = 1\}$ is not relatively open, because its relative boundary contains $(1/p^1, 0)$. This set could be relatively open if $w = (0, 1)$, in which case it would be $\{p \in \boldsymbol{R}^2 \mid p^1 > 0 \text{ and } p^2 = 1\}$. But this set does not contain any positive multiple of $p = (1, 0)$.

3 Existence of a Numeraire Vector

Recall that the vector w have two properties in the theorem. First, an equilibrium price vector, if any, is assumed to give a positive value to the commodity vector w. Second, if the price vector equals w, then the excess demands must be well defined.

In the case of strongly monotone preference relations, w can be taken to be $(1, 1, \ldots, 1)$ and V can be taken to be the singleton $\{w\}$. Then the boundedness from below and the standard boundary behavior condition, $\|z_n\| \to \infty$ as $n \to \infty$, where $z_n \in \zeta(p_n)$ and $(p_n)_n$ is an escaping sequence of strictly positive price vectors, implies Condition 4 of Theorem 1. For some cases of non-monotone preference relations, w may be taken to be $w = (1, 0, \ldots, 0)$ or even involving some negative coordinates.

We now show that there is no essential loss of generality in normalizing prices so that some vector, under which the aggregate excess demand exists, serves as the numeraire. For this result, we need to be explicit on conditions for the consumers' consumption sets and preference relations.

Proposition 2 *Suppose that the consumers' consumption sets are $\boldsymbol{R}^L_+$ and preference relations are complete, transitive, continuous, and globally non-satiated. Let $P \subset \boldsymbol{R}^L$ be the set of all price vectors under which the aggregate excess demands exist. If P is convex, then there exists a $w \in P$ such that $p \cdot w > 0$ for every $p \in P$.*

Proof of Proposition 2. By the completeness, transitivity, and continuity, $P \supseteq \boldsymbol{R}^L_{++}$.

Suppose that there exists a $p \in P$ such that $\{p, -p\} \subseteq P$. Then, for every consumer a, there exists a utility maximizing consumption vector for each of the two budget sets, $\{x \in \boldsymbol{R}^L_+ \mid p \cdot x \leq p \cdot e(a)\}$ and $\{x \in \boldsymbol{R}^L_+ \mid (-p) \cdot x \leq (-p) \cdot e(a)\} = \{x \in \boldsymbol{R}^L_+ \mid p \cdot x \geq p \cdot e(a)\}$. By transitivity, at least one of the two vectors is a utility-maximizing consumption vector on the entire $\boldsymbol{R}^L_+$. But this contradicts the global non-satiation property. Hence if $p \in P$, then $-p \notin P$.

By homogeneity, $P \cup \{0\}$ is a cone. Since P is convex and includes $\boldsymbol{R}^L_{++}$, $P \cup \{0\}$ is a closed and convex cone with a nonempty interior. Since, as proved in the last paragraph, $-p \notin P$ for every $p \in P$, $P \cup \{0\}$ is a pointed cone, that is, it does not contain any line.

Now denote by $\operatorname{cl} P$ the closure of P, then $\operatorname{cl} P$ is a closed convex cone with a nonempty interior. It need not be pointed, so we denote by M the largest linear subspace included in P and by N the orthogonal complement of M. Then $\operatorname{cl} P = M + (N \cap \operatorname{cl} P)$ and $N \cap \operatorname{cl} P$ is a pointed cone with a nonempty interior in N. By the separating hyperplane theorem being applied within N, there exists a $w' \in N$ such that $p \cdot w' > 0$ for every $p \in N \cap \operatorname{cl} P$ with $p \neq 0$. Then let $w'' \in N$ be the orthogonal projection of w' onto $N \cap \operatorname{cl} P$. Then $p \cdot w'' \geq p \cdot w'$ and hence $p \cdot w'' > 0$ for every $p \in N \cap \operatorname{cl} P$ with $p \neq 0$. Since $N \cap \operatorname{cl} P$ is pointed, there exists a $\delta > 0$ such that $p \cdot w'' > \delta$ for every $p \in N \cap \operatorname{cl} P$ with $\|p\| = 1$. Thus there exists a $w \in N$, sufficiently close to w'', such that w belongs to the interior of $N \cap \operatorname{cl} P$ in N, and hence of $N \cap P$ by the convexity of P, and $p \cdot w > \delta/2$ for every $p \in N \cap \operatorname{cl} P$ with $\|p\| = 1$. Thus $w \in P$ and $p \cdot w > 0$ for every $p \in P$. ///

4 Other Types of Boundary Behaviors

The condition of the boundary behavior that we imposed in Theorem 1 is that there exist a nonempty and compact subset V of P such that the set $\left\{p \in P \mid \min_{z \in \zeta(p)} \max_{v \in V} v \cdot z < 0\right\}$ is included in a compact subset of P. In this section we give two sufficient conditions for this one that are more along the lines of the existing literature. The result is the following.

Proposition 3 *If V is a nonempty and compact subset of P and the set $\left\{p \in P \mid \min_{z \in \zeta(p_n)} \max_{v \in V} v \cdot z \leq 0\right\}$ is compact, then the set $\left\{p \in P \mid \min_{z \in \zeta(p)} \max_{v \in V} v \cdot z < 0\right\}$ is compact as well.*

This proposition follows immediately from the fact that $\left\{p \in P \mid \min_{z \in \zeta(p)} \max_{v \in V} v \cdot z < 0\right\}$ is a subset of $\left\{p \in P \mid \min_{z \in \zeta(p)} \max_{v \in V} v \cdot z \leq 0\right\}$.

The second sufficient condition for the boundary behavior of Theorem 1 is in terms of sequences of price vectors.

Definition 4 Let P be a subset of H and let $(p_n)_n$ be an arbitrary sequence in P. Then $(p_n)_n$ is *escaping in P* if for every compact subset C of P there exist an N such that $p_n \notin C$ for every $n > N$.

According to this definition, the elements of an escaping sequence in P would eventually go outside any compact set. To give the idea of what escaping sequences are like, it would be helpful to give some preliminary results.

Lemma 5 *Let P be a subset of H and let $(p_n)_n$ be an arbitrary sequence in P.*

1. *If $(p_n)_n$ is divergent, that is, $\|p_n\| \to \infty$ as $n \to \infty$, then $(p_n)_n$ is escaping in P.*
2. *Suppose that $(p_n)_n$ converges to some $p \in H$ (with respect to the topology of the entire H). Then $(p_n)_n$ is escaping in P if and only if $p \notin P$.*
3. *Suppose that P is a bounded and open subset of H. Then $(p_n)_n$ is escaping if and only if all of its cluster points belong to the (relative) boundary of P.*

Proposition 6 *If V is a non-empty and compact subset V of P and, for every escaping sequence $(p_n)_n$ in P,*

$$\limsup_{n\to\infty} \min_{z\in\zeta(p_n)} \max_{v\in V} v \cdot z > 0, \tag{3}$$

then the set $\left\{p \in P \mid \min_{z\in\zeta(p_n)} \max_{v\in V} v \cdot z \le 0\right\}$ is compact.

By this proposition and Proposition 3, we can conclude that the condition (3) is sufficient for the boundary condition of Theorem 1.

Proof of Proposition 6. Denote by P^* the set $\left\{p \in P \mid \min_{z\in\zeta(p_n)} \max_{v\in V} v \cdot z \le 0\right\}$ and let $(p_n)_n$ be a sequence in P^*. It suffices to show that $(p_n)_n$ has a subsequence convergent in P^*. For each n, let $z_n \in \zeta(p_n)$ satisfy $\max_{v\in V} v \cdot z_n \le 0$.

If $(p_n)_n$ is an escaping sequence, then Condition 4 of Theorem 1 implies that $\min_{z\in\zeta(p_n)} \max_{v\in V} v \cdot z > 0$ for infinitely many n. But this contradicts the hypothesis that $(p_n)_n$ is a sequence in P^*. Hence $(p_n)_n$ is not an escaping sequence. Thus, there exist a compact set $C \subset P$ and a subsequence $(p_{k_n})_n$ such that $p_{k_n} \in C$ for every n. For each n, let $z_{k_n} \in \zeta(p_{k_n})$ satisfy $\max_{v\in V} v \cdot z_{k_n} \le 0$. Since ζ is convex- and compact-valued and upper hemi-continuous, by Theorem 1 of Hildenbrand (1974, I.B), there exists a subsequence $((p_{j_n}, z_{j_n}))_n$ of $((p_{k_n}, z_{k_n}))_n$ that converges to some $(p, z) \in C \times \boldsymbol{R}^L$ with $z \in \zeta(p)$. Then $\max_{v\in V} v \cdot z \le 0$ and hence $p \in P^*$. This completes the proof. ///

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