Heterogeneous Impatience of Individual Consumers and Decreasing Impatience of the Representative Consumer

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Abstract

In a continuous-time equilibrium model of heterogeneous consumers, we formulate and prove the statement that the more heterogeneous the consumers are in their impatience, the more dynamically consistent the representative consumer is. We apply this result to interest rate models, and, in particular, accommodate heterogeneous impatience in the model of Cox, Ingersoll, and Ross (1985) to come up with a new form of short-rate processes.

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1 Introduction

1.1 Background and motivation

Asset transactions are often motivated by the heterogeneity in consumers' characteristics. More risk-averse consumers unload their risk exposures and less risk-averse ones take them over with premiums; and more patient consumers save more to enjoy higher consumptions in the future and impatient ones borrow to enjoy higher consumption immediately. The raison d'être of asset markets is precisely to cater for diverse needs for asset transactions by heterogeneous consumers.

Heterogeneity in consumers' characteristics have implications not only on risk and intertemporal allocations but also on asset prices. The impact on asset prices can probably be best understood by constructing the representative consumer. The representative consumer is a fictitious consumer whose marginal utility process, evaluated along the average consumption process, is a state price deflator. For example, the representative consumer of individual consumers having utility functions of constant and unequal relative risk aversion has a utility function of strictly decreasing relative risk aversion (Franke, Stapleton, and Subrahmanyam (1999) and Hara, Huang and Kuzmics (2007)); and the representative consumer of individual consumers having constant but unequal subjective discount rates has discount rates that are a strictly decreasing function of time (Weitzman (2001), Gollier and Zeckhauser (2005), and Lengwiler (2005)). The consequences of these are that the derivative assets with convex payoff functions is underestimated if the coefficients of constant relative risk aversion are erroneously assumed to be equal (Franke, Stapleton, and Subrahmanyam (1999) and Hara, Huang, and Kuzmics (2007)); and that the term structure of interest rates is more downward sloping in the case of heterogeneous subjective discount rates than in the case of homogeneous subjective discount rates (Lengwiler (2005)).

Weitzman (2001), Gollier and Zeckhauser (2005), and Lengwiler (2005) showed that the heterogeneity in individual consumers' subjective discount rates gives rise to a decreasingly impatient representative consumer, that is, the representative consumer's discount rate is decreasing over time. But, they did not investigate whether a more heterogeneous economy give rise to a more decreasingly impatient representative consumer. The purpose of this paper is to give a precise formulation to the statement that the more heterogeneous the subjective discount rates of the individual consumers are, the more decreasingly impatient the representative consumer is. Doing so will enable us to extend the line of thoughts pursued especially by Gollier and Zeckhauser (2005) to fill in the spectrum of various degrees of individual consumers' heterogeneous impatience and the corresponding representative consumer's decreasing impatience.

1.2 Overview of the results of this paper

To formalize the idea that the more heterogeneous individual consumers' impatience leads to the more decreasingly impatient representative consumer, we need to give a notion of the "more decreasingly impatient than" relation and the "more heterogeneous than" relation, between two heterogeneous economies. For the former, we use the notion by Prelec (2004), which has the behavioral implication that a more decreasingly impatient consumer has preference reversal between two alternatives whenever the less decreasing impatient consumer does.[1]

The natural candidate for the latter is the mean-preserving spread (the second-order stochastic dominance with equal mean). That is, we might say that the individual consumers' impatience is more heterogeneous in an economy than in another if the (approximately wealth-weighted) distribution of the individual consumers' subjective discount rates of an economy is a mean-preserving spread of that of the other.[2] As we will give an example in Section 6.3, however, the mean-preserving spread does not always lead to the more decreasingly impatient representative consumer. The reason can be intuitively explained with the logic that can be traced

[1]Since, as Lengwiler (2005, Section IV) pointed out, the representative consumer is a fictitious agent, who does not autonomously choose asset portfolios but is used only to derive equilibrium state prices, this behavioral implication is not applicable to the representative consumer. The individual consumers in our model are, on the other hand, all constantly impatient and, thus, have no preference reversal.

[2]Since the representative consumer is defined so that his marginal utility coincides with the state-price density, the relevant distribution of the individual consumers' subjective discount rates is not the unweighted one, but the one weighted by their wealth. In the introduction and the subsequent analysis, we, in fact, consider approximately wealth-weighted distributions, of which the precise definition is given by (7).

back to Becker (1980) and Rader (1981, Section 6): As time goes to infinity, the most patient individual consumer's share in the average consumption and wealth converges to one. Moreover, the convergence of his share is faster, the more spread the individual consumers' discount rates are. Thus, if an economy consists, at the initial time 0, of more heterogeneous individual consumers, then the representative consumer is, for a while, more decreasingly impatient; but as time elapses, the most patient consumer's consumption and wealth share may increase faster, and the distribution of discount rates, weighted by the wealth shares on intermediate times, may become less heterogeneous, resulting in a less decreasingly impatient representative consumer. That is, the mean-preserving spread cannot rule that possibility that the ranking of decreasing impatience is switched as time elapses. It is, therefore, necessary to employ a more stringent mathematical definition for the more-heterogeneously-impatient-than relation to prevent the ranking of heterogeneity from switching as time elapses, even when the heterogeneity is assessed in terms of the distribution of discount rates, weighted by the consumers' wealth on any intermediate period.

This more stringent mathematical for the more-heterogeneously-impatient-than relation can be explained as follows. Since any distribution of individual consumers' discount rates is concentrated on the non-negative part $\boldsymbol{R}_+$ of the real line, we can define its cumulant-generating function, the logarithm of its moment-generating function, on the non-positive part $-\boldsymbol{R}_+$. Then our main result (Theorem 2 in 5) shows that the more convex the cumulant-generating function of the distribution of discount rate is, the more decreasingly impatient the representative consumer is, and the converse also holds.

To understand what the convexity assumption means, recall that the convexity of a twice differentiable function is measured by its curvatures, which are the ratios of the second derivatives to the first derivatives. It is well known that the first and second derivatives of the cumulant-generating function at any $s \leq 0$ coincide with the mean and variance of the probability measure of which the density function with respect to the distribution of discount rates is proportional to the exponential function $q \mapsto \exp(sq)$. Hence the curvature of the cumulant-generating function at s is the ratio of the variance to the mean of the probability measure with a density proportional to $q \mapsto \exp(sq)$. The curvature at 0 is equal to the ratio of the variance to the mean of the distribution. As such, it merely measures the heterogeneity of the discount rates at time 0. But, as we discussed in the preceding paragraph, we wish to prevent the ranking of decreasing impatience from switching at any future point in time. To do so, we need to impose the curvature condition at any future point in time. The resulting condition is that the curvature of the cumulant-generating function is larger for one distribution than for the other, at every $s \leq 0$, or, equivalently, the cumulant-generating function of one distribution is more convex than that of the other.

In Section 7, we apply the main result, and its weaker variant to be given in Section 5, to the analysis of the term structure of interest rates. Our approach is rather unique. While the literature on the term structure of interest rates typically looks into the conditions under which, say, the yield curve of an economy is flat, upward sloping, or downward sloping, we take up two economies having a common average (aggregate) consumption process but differing heterogeneity in individual consumers' impatience and ask whether, say, the yield curve of one economy is more upward or downward sloping than that of the other. As for the short-rate processes, we show that the most commonly used structures are invariant under changes in the heterogeneity in individual consumers' impatience. More specifically, we prove that if the short-rate process of one economy is Gaussian or has the affine structure, then that of the other economy is also Gaussian or has the affine structure. We extend the short-rate process in the model of Cox, Ingersoll, and Ross (1985) to the case in which the representative consumers has generalized hyperbolic discounting of Lowenstein and Prelec (1992). Although the resulting short-rate process still has the affine structure, its mean, to which the process is reverting, turns out to be time-varying.

1.3 Related literature

Lengwiler (2005) considered the term structure of interest rates in a deterministic model of discrete time of finite span populated by finitely many consumers sharing the logarithmic utility function and the same wealth level. He evaluated the impact of heterogeneous impatience on the term structure of interest rates by comparing such a heterogeneous economy with a homogeneous economy, rather than with another, less heterogeneous one. In addition, the benchmark homogeneous economy he compared with the given heterogeneous economy is the one in which all consumers have the discount factor, not the discount rate as we will do in this paper, that is equal to the mean of the individual consumers' counterparts. The risk-free rate (short rate) at the initial date (time 0) of the benchmark homogeneous economy is equal to that of the original heterogeneous economy in our benchmarking (where the discount rate of the homogeneous economy is equal to the mean discount rate of the heterogeneous economy), while the former is lower than the latter in his benchmarking (where the discount factor of the homogeneous economy is equal to the mean discount factor of the heterogeneous economy), due to Jensen's inequality.

In a series of works, Rohde (2009, 2010, 2018) introduced a variety of measures of decreasing impatience, some of which were used to explain experimental finding. We will comapre Prelec's measure of decreasing impatience, which we employ throughout this paper, with these measures in detail towards the ends of Section 4 and, also, Section 5.

In the literature, there have been two important areas of research that has attracted attention to decreasing impatience. One is the possibility of aggregating heterogeneous impatience, in the context of social choice, into a single preference while respecting unanimity and time consistency without giving rise to dictatorship. The other is the impact of time inconsistency and decreasing impatience on the solution to an optimal stopping-time problem. Although these problems are not addressed in this paper, let us mention some contributions on these questions in turn to illustrate the analytical affinity of this paper with theirs.

On the possibility of aggregating heterogeneous impatience, Zuber (2011) showed, in a discrete-time model in which a social welfare function is defined on the set of profiles of individual consumption processes and each individual agents' utility functions are defined on the set of temporal lotteries in the sense of Epstein and Zin (1989), that the social welfare function satisfies unanimity, time consistency, and stationarity only if the individual agents have constant and common discount factor. In cases in which a common consumption process is shared among the individual agents, Jackson and Yariv (2014) showed that if their utility functions are aggregated into a social welfare function in a utilitarian manner, then the associated discount factor function must necessarily exhibit decreasing impatience and, thus, time inconsistency. Jackson and Yariv (2015) showed further that the time inconsistency would still prevail, were the dictatorship to be avoided, when the individual agents' utility functions are aggregated, in any way that respects unanimity, into a time-additive social welfare function with the multiplicative form of discount factors and instantaneous utilities, or even when the social ranking is, at the outset, not assumed to be representable by such a social welfare function.

As the representative consumer in this paper has a time-additive, multiplicatively separable utility function with decreasing impatience, it does in no way contradict their results. But it cannot really be regarded as representing the social welfare function in these papers for the following reasons. First, at the conceptual level, the aggregation is done in our paper through markets, in the sense that the equilibrium prices determine the utility weights based on which the utilitarian welfare maximization problem is defined and solved.[3] As such, the representative consumer's discount factor function is merely a reflection of the state price density or stochastic discount factor, with no deliberate respect for unanimity or time consistency. Moreover, the representative consumer in this paper consumes the sum of what the individual consumers consume, and the individual consumptions have one-to-one relations to the aggregate consumption via the solution to the utilitarian welfare maximization problem.[4] As such, a change in aggregate consumptions has heterogeneous impacts on individual consumptions.

Chambers and Echenique (2018a) took up a similar situation, where heterogenous agents are interpreted as experts whose task is to evaluate the intergenerational welfare consequences of long-run projects. They investigated under what axioms the social ranking can be represented by a function of a form that are reminiscent of the maximin utility of Gilboa and Schmeidler (1989), where there is a set Σ of probability distributions on a set of discount factors that agents might have (a subset of an open unit interval $(0,1)$) and the function attaches to each intergenerational utility stream the minimum of the values of the weighted utilitarian social welfare based on the probability distributions in Σ. Such a social ranking embodies the criterion that they referred to as multi-utilitarianism, and necessarily violates time-consistency. The discount factor function in our setting, defined by (4), can be considered as a consequence of aggregating heterogeneous impatience, though without axiomatization but through asset markets, is just between the two polar cases, utilitarian and maximin, of their multi-utilitarianism. Chambers and Echenique (2018b) took up the same setting but investigated under what axioms the social ranking on the set of intergenerational utility streams represents the unanimous agreement of agents with constant, but no common,

[3]The function λ and the maximization problem (2) in Section 2.2.

[4]A consequence of this is that when speaking of utilitarian aggregation, Jackson and Yariv (2014, 2015) took a weighted sum of utility functions, while we take a weighted sum of the Legendre transforms of utility functions. Karatzas and Shreve (1998, Section 4.5) explain the use of Legendre transforms in equilibrium analysis.

discount rates. They also obtained the dual expression of the utility streams that can be unanimously preferred to a given one. The expression is derived from the inversion formula of Bernstein's Theorem, which we will mention in Section 6.2.

Feng and Ke (2018) avoided the equivalence between the combination of time consistency and unanimity on the one hand and dictatorship on the other, by introducing altruism into individual agents' utility functions. Specifically, they assumed that each individual agent lives for just one period but cares about the future agents' welfare, while discounting it by a constant discount rate. Thus, for every period t, the consumption of the agent on period t enters into all agents' utility functions up to and including period t. Because of this altruism, the (constant) discount factor embedded in a social ranking satisfying time consistency and unanimity may well be (and, under some conditions, must be) higher than all individual agents' counterparts for the social ranking to respect unanimity, which would not be possible in the setting of Jackson and Yariv (2014, 2015).

As for the impact of time inconsistency and decreasing impatience on stopping time, Quah and Strulovici (2014) studied the impact of the decision maker's impatience on his optimal choice in a stopping time problem. They defined a decision maker's discount factor function as at least as patient as another if they have the monotone likelihood ratio property, and proved that under general condition that admit stochastic terminal payoffs, the possibility of controlling intermediate flow payoffs, stochastic discount factors, and decreasing impatience, it is optimal for a more patient decision maker to enjoy higher intermediate flow payoffs and terminate the process later, thereby attaining a higher value function. They applied this result (in Section 5 of their paper) to a decision maker who exhibits decreasing impatience. The application relies partly on the assumption of decreasing impatience, but their focus is on the impact of the degree of impatience itself on the optimal choice of a single decision maker's problem, while our focus is on the impact of the heterogeneity of individual consumers' impatience on the degree of decreasing impatience of the representative consumer.

Ebert, Wei, and Zhou (2018) studied how the increase in the heterogeneity of individual consumers' impatience affect on the collective choice in the stopping-time problem. To measure the heterogeneity, they a generalized version of eventual dominance of Fishburn (1980, Section 4) and showed that the more heterogeneous the individual consumers' impatience, the later the group chooses to stop in the stopping time problem. We will compare Fishburn's measure of heterogeneity with our measure of heterogeneity towards the end of Section 3.

1.4 Organization of this paper

The setup and preliminary results are presented in Section 2. The measure of heterogeneity of impatience is introduced in Section 3. The more-decreasingly-impatient-than relation of Prelec (2004) is reviewed in Section 4. The main results are stated in Section 5. Parameterized examples of the main results are given in Sections 6. Applications to the term structure of interest rates are explored in Section 7. The results are summarized and a future research topic is suggested in Section 8. Appendix A lays out an equilibrium foundation of our approach of comparing two economies with a common average endowment process but differing heterogeneity of individual consumers' impatience. Appendix B gathers proofs.

2 Setup and Preliminary Results

In this section, we define an economy as a collection of individual consumers with heterogeneous discount rates and derive the discount rates of the representative consumer of the economy.

2.1 Economy

The economy is subject to uncertainty, which is represented by a probability space $(\Omega, \mathscr{F}, P)$. The time span is $\boldsymbol{R}_+ = [0, \infty)$, which is of continuous time. Its length is assumed to be infinite, but could be taken to be finite, such as $[0, T]$ with $T < \infty$, as the subsequent result does not depend on the length of time span. The gradual information revelation is represented by a filtration $(\mathscr{F}(t))_{t \in \boldsymbol{R}_+}$. There is only one type of good on each time and state.

We allow the number of consumers present in the economy to be finite or infinite. Formally, we let $(I, \mathscr{I}, \iota)$ be a probability space of (names of) consumers. It is customary to take I be the unit interval $(0, 1)$, $\mathscr{I}$ be the σ-field of Lebesgue measurable subsets of I, and ι be (the restriction of) the Lebesgue measure on $\mathscr{I}$, and, then, the consumption sector consists of infinitely many consumers, each of whom is negligible in size relative to the total population of the economy. For each $J \in \mathscr{I}$, $\iota(J)$ is the proportion of consumers in J relative to the entire consumption sector.

We assume that the consumers have time-additive expected utility functions over consumption processes, which exhibit constant and equal relative risk aversion, and constant but possibly unequal discount rates. Formally, let $\gamma > 0$ and define $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by

$$u(x) = \begin{cases} \ln x & \text{if } \gamma = 1, \\ \dfrac{x^{\gamma-1} - 1}{\gamma - 1} & \text{otherwise,} \end{cases}$$

for every $x \in \boldsymbol{R}_{++}$. Let $\rho : I \to \boldsymbol{R}_{++}$ specify the individual consumers' discount rates. Then the utility function U_i of consumer i over consumption processes is defined by

$$U_i(c_i) = E\left(\int_0^\infty \exp(-\rho(i)t)u(c_i(t))\,\mathrm{d}t\right),$$

where $c_i = (c_i(t))_{t\in\boldsymbol{R}_+}$.[5] Although the assumption of constant and equal relative risk aversion is quite stringent, there is a good reason to restrict our attention to this case. In fact, Hara (2009, Corollary 2) showed that if consumers had unequal coefficients of constant relative risk aversion, then the representative consumer's discount factor function, to be specified below, would not be well defined.

For each $i \in I$, let e_i be the endowment process of consumer i, which is an $\boldsymbol{R}_+$-valued adapted process. The average (mean) endowment process, $\int_I e_i \,\mathrm{d}\iota(i)$, is denoted by e.

2.2 Arrow-Debreu equilibrium

A *state-price deflator* is an $\boldsymbol{R}_{++}$-valued adapted process. The utility maximization problem of consumer i under a state-price deflator π is

$$\begin{aligned} \max_{c} \quad & U_i(c_i) \\ \text{subject to} \quad & E\left(\int_0^\infty \pi(t)(c(t) - e_i(t))\,\mathrm{d}t\right) = 0. \end{aligned} \tag{1}$$

We say that a state price deflator π and an allocation $(c_i)_{i\in I}$ of consumption processes constitute an *Arrow-Debreu equilibrium* if $\int_I c_i \,\mathrm{d}\iota(i) = e$ and for every $i \in$

[5]This and other integrals in the subsequent analysis need not be well defined without additional assumptions on c_i and other stochastic processes. But the subsequent argument depends only on the first-order conditions of (utility or social welfare) maximization problems, which must necessarily hold whenever there is a solution to the problem under consideration. We shall therefore be implicit about these additional assumptions.

I, c_i is a solution to the utility maximization problem (1) of i under the state-price deflator π. Although we shall not elaborate on this point, it is well known that the Arrow-Debreu equilibrium allocations coincide with the equilibrium allocations in complete asset markets.

2.3 Representative consumer

By Proposition 10.C of Duffie (2001), for each Arrow-Debreu equilibrium, there is a $\lambda : I \to \boldsymbol{R}_{++}$ such that the solution to the social welfare maximization problem over allocations of the average consumption process c,

$$\begin{aligned} \max_{(c_i)_{i \in I}} \quad & \int_I \lambda(i) U_i\left(c_i\right) \,\mathrm{d}\iota(i), \\ \text{subject to} \quad & \int_I c_i \,\mathrm{d}\iota(i) = c. \end{aligned} \tag{2}$$

coincides with the equilibrium allocation when $c = e$.[6] It was shown in Hara (2008, Section 2) that the value function of this maximization problem, which is the representative consumer's utility function, is given by

$$U(c) = E\left(\int_0^\infty f(t) u(c(t)) \,\mathrm{d}t\right). \tag{3}$$

where

$$f(t) = \left(\int_I \left(\lambda(i) \exp(-\rho(i)t)\right)^{1/\gamma} \,\mathrm{d}\iota(i)\right)^\gamma. \tag{4}$$

Note that the representative consumer too has constant relative risk aversion equal to γ. The function $d : \boldsymbol{R}_+ \to \boldsymbol{R}$ is the representative consumer's *discount factor function.*[7] In order for U to be well defined, it is necessary and sufficient that $d(0) < \infty$, because $\exp(-\rho(i)t) \leq 1$ for every $i \in I$ and every $t \in \boldsymbol{R}_+$. This is equivalent to saying that the function $i \mapsto (\lambda(i))^{1/\gamma}$ is integrable with respect to

[6] To be precise, Proposition 10.C of Duffie (2001) is valid when there are only finitely many consumers. If there are infinitely many consumers, then the utility possibility set, the set of vectors of the consumers' utility levels at the feasible allocations, is a subset of the set of the real-valued functions on the set I of consumers, which is an infinite-dimensional vector space. Under the assumptions mentioned in Footnote 5, the utility possibility set is convex, and, if, in addition, the function $i \mapsto |U_i(c_i)|$ is bounded, then the Hahn-Banach theorem (a separating hyperplane theorem in infinite-dimensional vector spaces) guarantees the existence of such a $\lambda : I \to \boldsymbol{R}_{++}$.

[7] Weitzman (2001) and Ebert, Wei, and Zhou (2018) considered the case where $\gamma = 1$ and λ is constantly equal to one.

ι. As we will see, d is an analytic function. Define the representative consumer's *discount rate function* $r : \boldsymbol{R}_+ \to \boldsymbol{R}_{++}$ by

$$r(t) = -\frac{f'(t)}{f(t)}, \tag{5}$$

then

$$\frac{f(t_2)}{f(t_1)} = \exp\left(-\int_{t_1}^{t_2} r(t)\,\mathrm{d}t\right)$$

whenever $0 \leq t_1 < t_2$. Thus r represents the representative consumer's continuously compounded instantaneous subjective discount rate as a function of time. Unlike the case of individual consumers, this is not constant but varies with t unless all individual consumers have the same discount rate.

The equilibrium state price deflator is given by $(f(t)u'((e(t))))_{t\in\boldsymbol{R}_+}$. Thus the price at time t_1, relative to the current consumption, of the discount bond with maturity $t_2 > t_1$ is equal to

$$E_{t_1}\left(\frac{f(t_2)u'(e_{t_2})}{f(t_t)u'(e_{t_1})}\right) = \frac{f(t_2)}{f(t_1)}E_{t_1}\left(\frac{u'(e_{t_2})}{u'(e_{t_1})}\right) = \exp\left(-\int_{t_1}^{t_2} r(t)\,\mathrm{d}t\right)E_{t_1}\left(\frac{u'(e_{t_2})}{u'(e_{t_1})}\right).$$

In Section 7, we will see how the heterogeneity in ρ affect these bond prices and associated interest rates.

3 Measure of heterogeneous impatience

In this section, we use cumulant-generating functions to introduce a measure of heterogeneity in the individual consumers' discount rates. Recall from (4) that the representative consumer's discount factor function d can be written as

$$f(t) = \left(\int_I (\lambda(i))^{1/\gamma} \exp\left(-\frac{\rho(i)t}{\gamma}\right) \mathrm{d}\iota(i)\right)^{\gamma}. \tag{6}$$

Define a probability measure μ on $\boldsymbol{R}_{++}$ by letting

$$\mu(B) = \left(\int_I (\lambda(i))^{1/\gamma}\,\mathrm{d}\iota(i)\right)^{-1} \int_{\rho^{-1}(B)} (\lambda(i))^{1/\gamma}\,\mathrm{d}\iota(i) \tag{7}$$

for every Borel-measurable subset B of $\boldsymbol{R}_{++}$. Alternatively, μ can be defined by first letting ι_λ the finite measure on I for which

$$\frac{\mathrm{d}\iota_\lambda}{\mathrm{d}\iota} = \left(\int_I (\lambda(i))^{1/\gamma}\,\mathrm{d}\iota(i)\right)^{-1} \lambda^{1/\gamma}$$

and then letting $\mu = \iota_\lambda \circ \rho^{-1}$. By definition, for every Borel-measurable subset B of $\boldsymbol{R}_{++}$, $\mu(B)$ is the fraction of the consumers whose discount rates are in B, where the fraction is calculated from the probability measure ι on the population I and the density function on I that is proportional to $\lambda^{1/\gamma}$. Denote its moment-generating function by M, that is, $M(s) = \int_{\boldsymbol{R}_{++}} \exp(sq)\,\mathrm{d}\mu(q)$. Then

$$\begin{aligned} f(t) &= \left(\int_I (\lambda(i))^{1/\gamma}\,\mathrm{d}\iota(i)\right)^{\gamma} \left(\int_I \exp\left(-\frac{qt}{\gamma}\right)\mathrm{d}\mu(q)\right)^{\gamma} \\ &= \left(\int_I (\lambda(i))^{1/\gamma}\mathrm{d}\iota(i)\right)^{\gamma} \left(M\left(-\frac{t}{\gamma}\right)\right)^{\gamma}. \end{aligned} \tag{8}$$

Denote its cumulant-generating function by K, that is, $K(s) = \ln M(s)$. Then

$$r(t) = K'\left(-\frac{t}{\gamma}\right). \tag{9}$$

It is well known if the first two moments exist (are finite), then K is twice differentiable, with $K'(0)$ equal to the mean of μ and $K''(0)$ equal to the variance of μ. More generally, denote by $\mu(s)$ the probability measure on $\boldsymbol{R}_{++}$ such that

$$\frac{\mathrm{d}\mu(s)}{\mathrm{d}\mu}(q) = \exp\left(sq - K(s)\right) \tag{10}$$

for every $q \in \boldsymbol{R}_{++}$. Then, by Morris (1982, Section 2), for every s, $K'(s)$ and $K''(s)$ are equal to the mean and variance of $\mu(s)$. Thus, $K'(s) > 0$ for every s, and, unless μ is concentrated on a single point, $K''(s) > 0$ for every s. Hence $K''(s)/K'(s)$ is the ratio of the variance to the mean, and can be considered as a measure of dispersion of the probability measure μ. We shall take it as the measure of heterogeneity of the individual consumers' discount rates.

Definition 1 A probability measure μ_1 on $\boldsymbol{R}_{++}$ is at least as heterogenous as another probability measure μ_2 if, for the corresponding cumulant-generating functions K_1 and K_2, $K_1''(s)/K_1'(s) \geq K_2''(s)/K_2'(s)$ for every $s \leq 0$.

Since $K''(s)/K'(s)$ measures the curvature of the function K, this means that the individual consumers' discount rates in an economy is more heterogeneous than in another if the cumulant-generating function of the distribution of discount rates is more convex for the first economy than for the second.

As can be seen in (6) and Definition 1, both the representative consumer's discount factor function and the at-least-as-heterogeneous-as relation depend on the utility weighting function λ, which has been chosen so that the solution to the social welfare maximization problem (2) is the Arrow-Debreu equilibrium allocation. This fact may be an impediment to applications of our subsequent results, because it is, in general, difficult to relate the consumers' utility weights to their wealth levels.[8] In Appendix A, we give some, albeit special, results on how the two are related to each other and how our results can be used to assess the impact of changes in consumers' impatience or wealth levels.

In the rest of this section, we mention other candidates for the measure of heterogeneity of the individual consumers' discount rates. The most natural candidate would be the mean-preserving spread, or, equivalently, the second-order stochastic dominance relation with a common mean, as it has been widely used in the theory of expected utility. As we explained in the introduction, however, this relation is not sufficient to guarantee the at-least-as-decreasingly-impatient-as relation, in the sense to be defined in Section 4.

Another candidate is the eventual dominance introduced by Fishburn (1980, Section 4). Roughly, one distribution eventually is said to dominate another if there is a positive integer n such that the former n-th-order stochastically dominates the latter. Then the equivalent condition of eventual dominance that Fishburn (1980, Corollary 2) established in the case of simple distributions (distributions that put positive probabilities only on finitely many points) can be interpreted in our case as saying that one distribution of the individual consumers' discount rates eventually dominates another if and only if the representative consumer's discount factor function derived from the first distribution takes a higher value than that derived from the second at every point in time. This result indicates that eventual dominance bears upon the levels of discount factors, not upon the rate of decrease of discount rates, the latter of which represents decreasing impatience. A simple example illustrates this point. Consider two homogeneous

[8]An exception is Leamma 4.1 of Jouini and Napp (2007), who gave bounds on the discrepancy between the two.

economies, of which all individual consumers have a higher (common) discount rate in the one economy than in the other. Then the distributions of discount rates in the two economies are both degenerate, and the distribution of the first economy first-order stochastically dominates that of the second. By definition, this implies that the distribution of the first economy eventually dominates that of the second. Yet, the two representative consumers are equally (in fact, not at all) decreasingly impatient as there is no heterogeneity in the two economies.[9] Ebert, Wei, and Zhou (2018) used a generalized version of eventual dominance and studied the optimal stopping time problems when the decision maker's discount factor function is an weighted average of negative exponential functions, as in (6), taking into consideration the effect of the higher values of discount factor functions arising from eventual dominance.

4 Measure of decreasing impatience

Prelec (2004) introduced an *more-decreasingly-impatient-than* relation between two utility functions over *single dated outcomes*, that is, functions of the form $f(t)u(x)$ defined over consumption levels x consumed at time t, where $f : \boldsymbol{R}_+ \to (0, 1]$ is strictly decreasing, and $u : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is strictly increasing and satisfies $u(0) = 0$. The following definition is a variant of the relation, is concerned with discount factor functions, rather than utility functions over timed consumptions.

Definition 2 A discount factor function f_1 is *at least as decreasingly impatient as* another discount factor function f_2 if for every $(t_0, t_1, t_2, \tau) \in \boldsymbol{R}_+^3 \times \boldsymbol{R}$,

$$\frac{f_2(t_0 + t_1)}{f_2(t_0)} \geq \frac{f_2(t_0 + t_1 + t_2 + \tau)}{f_2(t_0 + t_2)} \tag{11}$$

whenever

$$\frac{f_1(t_0 + t_1)}{f_1(t_0)} = \frac{f_1(t_0 + t_1 + t_2 + \tau)}{f_1(t_0 + t_2)}. \tag{12}$$

[9]A more complicated, but probably more relevant, example can be constructed from Gamma distributions in Subsection 6.1. Of the family of Gamma distributions parameterized by two parameters (α, β), every subfamily parameterized only by α, with any fixed values for β, can be linearly ordered by the first-order stochastic dominance, because the corresponding density functions satisfy the monotone likelihood ratio property. Thus, it is linearly ordered by eventual dominance as well. Yet, as we will see in the subsection, the representative consumers derived from the distributions of this subfamily are all equally decreasingly impatient.

To understand this definition, first compare the two ratios, $f_1(t_0+t_1)/f_1(t_0)$ and $f_1(t_0+t_1+t_2)/f_1(t_0+t_2)$. The former is the discount factor that f_1 applies to the time interval $[t_0, t_0+t_1]$, and the latter is the discount factor that d_1 applies to the time interval $[t_0+t_2, t_0+t_1+t_2]$. Since they are both applied to time intervals of length t_1, they would be equal if f_1 exhibited exponential discounting. However, they can be different, and, more specifically, the former is smaller than the latter if the corresponding discount rate function is decreasing over time, just as in the case of hyperbolic discounting. To compensate the difference between the two ratios, we add an interval of length τ (which is positive if the discounting rate function is decreasing over time, but negative if it is increasing) to the terminal time $t_0+t_1+t_2$ of the interval, so that the discount factor that f_1 applies to $[t_0+t_2, t_0+t_1+t_2+\tau]$ is equal to the discount factor that f_1 applies to $[t_0, t_0+t_1]$, as shown in (12). The length τ can, therefore, be considered as a measure of decreasing impatience of f_1. Then (11) states that τ may too large for f_2, so that the discount factor that f_2 applies to $[t_0+t_2, t_0+t_1+t_2+\tau]$ may be smaller than the discount factor that f_2 applies to the time interval $[t_0, t_0+t_1]$. In this sense, the impatience of f_1 decreases at least as rapidly as that of f_2 as the time interval under consideration is shifted into a more distant future. This is precisely the idea that Definition 2 embodies.

Prelec (2004, Proposition 1) proved the following equivalence on the at-least-as-decreasingly-impatient-as relation.

Theorem 1 (Prelec (2004)) *Let d_1 and d_2 be thrice differentiable discount factor functions and r_1 and r_2 be the corresponding discount rate functions. Then the following two conditions are equivalent.*

1. *f_1 is at least as decreasingly impatient as f_2.*
2. *$-r_1'(t)/r_1(t) \geq -r_2'(t)/r_2(t)$ for every $t \in \boldsymbol{R}_+$.*

By this theorem, we can use $-r_n'/r_n$ as the measure of decreasing impatience. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{r_1(t)}{r_2(t)}\right) = \frac{r_1(t)}{r_2(t)}\left(\frac{r_1'(t)}{r_1(t)} - \frac{r_2'(t)}{r_2(t)}\right),$$

the second is equivalent to the *monotone likelihood ratio property*, in that $r_1(t)/r_2(t)$ is a strictly decreasing function of t. Thanks to this theorem, to determine the ranking of the degree of decreasing impatience between two discount factor functions, it is sufficient to compare the rates of decrease of the corresponding discount rate functions. In the subsequent analysis, we identify how the rate of decrease of the representative consumer's discount rate function is related to the degree of heterogeneity of individual consumers' discount rates.

In the rest of this section, we mention three measures of decreasing impatience introduced by Rohde (2009, 2010, 2018). Rohde (2010) and Rohde (2018) introduced measures of decreasing impatience, termed, respectively, as the hyperbolic factor and the DI measure, that are applicable to utility functions not in the multiplicative form $f(t)u(x)$. They both depend on three points in time and one consumption level, and the ordering that they represent coincide with Prelec's (2004) more-decreasingly-impatient-than relation. The hyperbolic factor takes constant values for the generalized hyperbolic discount factor functions in the sense of Lowenstein and Prelec (1992, Section III). We will see this in Section 6.1. In the case of twice continuously differentiable discount factor functions, the DI measure converge to Prelec's measure $-r'(t)/r(t)$ of decreasing impatience as the difference between three time points converge to zero. The advantage of the DI measure over the hyperbolic factor is that the DI measure can measure increasing impatience and strongly decreasing impatience, in the latter case of which the hyperbolic factor would take negative values even when the discount factor function does indeed exhibit decreasing impatience.

Rohde (2009) introduced the notion of decreasing *relative* impatience. It refers to the tendency that a consumer is less willing to sacrifice the current consumption to speed up consumption at a future point in time, the further into the future the consumption occurs. She then introduced (Definition 3.3 of Rohde (2009)) the more-decreasingly-relatively-impatient-than relation, and proved (Theorem 4.1 of Rohde (2009)) that one discount factor function f_1 is more decreasingly relatively impatient than another discount factor function f_2 if and only if $-f_1''/f_1' \geq -f_2''/f_2'$, that is, the curvature of f_1 is everywhere higher than that of f_2.[10] The difference between decreasing relative impatience and decreasing impatience is that the former is measured by the curvature of the discount factor function but the latter is measured by the curvature of its logarithm (because $r_i = -(\ln f_i)'$). Since the former is concerned with two consumption time points (current and future) but the latter is concerned with just one point (future), it is potentially more useful to study the behavior of a consumer who can consume at any point in time on the

[10]This is equivalent to the monotone likelihood ratio property, where $f_1'(t)/f_2'(t)$ is a strictly decreasing function of t.

continuous time span $\boldsymbol{R}_+$. We shall, however, not dwell on the analysis for two reasons. First, since the curvature of a discount factor function can be written in terms of the moment-generating function of the distribution of the individual consumers' discount rates, the more-decreasingly-relatively-impatient-than relation can be characterized in terms of the derivatives of the moment-generating function,[11] and the analysis goes much in the same way as the analysis for the more-decreasingly-impatient-than relation of this paper. Second, as Theorem 4.5 of Rohde (2009) showed, the measure of decreasing relative impatience, $-f_i''/f_i'$, is affected by the size of the constant discount rate, it does not disentangle the effect of time variation of discount rates from that of the level of the discount rates.

5 Comparison between two economies

Consider two economies, $n = 1, 2$, each with the space $(I_n, \mathscr{I}_n, \iota_n)$ of (names of) consumers, a discount rate function $\rho_n : I_n \to \boldsymbol{R}_{++}$, and a weighting function $\lambda_n : I_n \to \boldsymbol{R}_{++}$, derived from (2). Assume that the consumers of the two economies share the same coefficient γ of constant relative risk aversion. Using $(I_n, \mathscr{I}_n, \iota_n)$, ρ_n, and λ_n, define the discount factor function $f_n : \boldsymbol{R}_+ \to \boldsymbol{R}_{++}$ in the same way as d in (4), and the discount rate function $r_n : \boldsymbol{R}_+ \to \boldsymbol{R}_{++}$ in the same way as r in (5). Define the probability measure μ_n on $\boldsymbol{R}_{++}$ in the same way as μ in (7), and let K_n be the cumulant-generating function of μ_n. These probability measures may have been derived under the assumption that and the two economies share the same space of consumers and the same average endowment process, and all consumers have the logarithmic utility function, as in Lemma 1 in Appendix A. It is possible but unnecessary to impose these assumptions for the results of this section.

By differentiating both sides of (9), we obtain

$$r_n'(t) = -\frac{1}{\gamma} K_n'' \left(-\frac{t}{\gamma} \right), \tag{13}$$

$$-\frac{r_n'(t)}{r_n(t)} = \frac{1}{\gamma} \frac{K_n'' \left(-\frac{t}{\gamma} \right)}{K_n' \left(-\frac{t}{\gamma} \right)}. \tag{14}$$

[11] In particular, if the constant relative risk aversion γ is equal to one, then the curvature of a discount factor function is equal to the curvature of the moment-generating function.

The following theorem follows immediately from these equalities.

Theorem 2 *The following two conditions are equivalent.*

1. *For every* $t \geq 0$, $-r_1'(t)/r_1(t) > -r_2'(t)/r_2(t)$.
2. *For every* $s \leq 0$, $K_1''(s)/K_1'(s) > K_2''(s)/K_2'(s)$.

The first condition of this theorem states that the representative consumer of the first economy is more decreasingly impatient than the representative consumer of the second economy. The second condition states that the cumulant-generating function of the distribution μ_n of individual consumers' discount rates in the first economy is more convex than in the second economy. This condition is equivalent to saying that the variance divided by the mean of the individual consumers' discount rates is higher in the first economy than in the second whenever their distribution is transformed by a negative exponential density function. The theorem, then, asserts that the representative consumer is more decreasingly impatient if and only if the cumulant-generating function of the distribution of individual consumers' discount rates is more convex.

The next theorem deals with weaker conditions, although they are still useful to investigate the term structure of interest rates.

Theorem 3 *The following two conditions are equivalent.*

1. *For every* $t \geq 0$, *if* $r_1(t) = r_2(t)$, *then* $r_1'(t) < r_2'(t)$.
2. *For every* $s \leq 0$, *if* $K_1'(s) = K_2'(s)$, *then* $K_1''(s) > K_2''(s)$.

The condition in the first part of this theorem implies the single-crossing property, in that r_1 crosses r_2 at most once from above: if $r_1(t_0) = r_2(t_0)$, then $r_1(t) < r_2(t)$ for every $t > t_0$, and $r_1(t) > r_2(t)$ for every $t < t_0$. That is, the discount rate in the first economy is higher than in the second up to a time, after which the former is lower. The second part is the single-crossing property of the K_n', where K_1' crosses K_2' at most once from below. This condition is equivalent to saying that if the distributions of the individual consumers' discount rates are transformed by a negative exponential density function so that the means after this transformation are equal in the two economies, then the variance is higher in the first economy than in the second. It is weaker than the monotone likelihood ratio property in Theorems 1 and 2. This theorem follows directly from (9) and (13).

In rest of this section, we see what kind of results, analogous to Theorem 2, would be obtained for other definitions of more-(decreasingly-)impatient-than relation. Of three relations we shall take up, let us first consider the higher discount factors, or, the more-elevated-than relation, of Fishburn (1980, Corollary 2) and Eber, Wei, and Zhou (2017). This relation means, in symbols, that $f_1(t)/f_1(0) \geq f_2(t)/f_2(0)$ for every $t \geq 0$.[12] By (8), $f_1(t)/f_1(0) \geq f_2(t)/f_2(0)$ for every t if and only if $K_1(s) \geq K_2(s)$ for every $s \leq 0$. That is, one representative consumer has higher discount factors than another if and only if the former is derived from the weight distribution of the individual consumers' discount rates of which the cumulant-generating function takes higher values everywhere than that of the latter. Second, consider the more-patient-than relation of Quah and Strulovici (2014, Definition 1). In symbols, this relation means that $f_1(t)/f_2(t)$ is a non-decreasing function of t. In case the f_n are differentiable, this is equivalent to saying that $r_1(t) \leq r_2(t)$ for every $t \geq 0$. By (9), $r_1(t) \leq r_2(t)$ for every t if and only if $K_1'(s) \leq K_2'(s)$ for every $s \leq 0$. Thus, one representative consumer is more patient than another if and only if the former is derived from the distribution of the individual consumers' discount rates of which the cumulant-generating function has higher slopes everywhere than that of the latter. Thus, these two relations of more (decreasing) impatience are concerned with the values and first derivatives of the cumulant-generating function of the distribution of individual consumers' discount rates, while Prelec's (2004) more-decreasingly-impatient-than relation, which was considered in Theorem 2, has to do with the curvature of the cumulant-generating function.

The last relation we consider is the more-decreasingly-relatively-impatient-than relation of Rohde (2009, Definition 3.2). In symbols, this means that $-f_1''(t)/f_1'(t) \geq -f_2''(t)/f_2'(t)$ for every $t \geq 0$. By (9) and (13),

$$-\frac{f_n''(t)}{f_n'(t)} = r_n(t) - \frac{r_n'(t)}{r_n(t)} = K_n'\left(-\frac{t}{\gamma}\right) + \frac{1}{\gamma}\frac{K_n''\left(-\dfrac{t}{\gamma}\right)}{K_n'\left(-\dfrac{t}{\gamma}\right)}. \tag{15}$$

Thus, one representative consumer is more decreasingly relatively impatient than another if and only if the cumulant-generating functions K_1^1 and K_2^1 of the distri-

[12]Normalization by dividing by $f_n(0)$ is necessary because $f_n(0)$ need not be equal to 0 according to the definition (6) of f_n.

butions of the individual consumers' discount rates satisfy

$$\gamma K_1'(s) + \frac{K_1''(s)}{K_1'(s)} \geq \gamma K_2'(s) + \frac{K_2''(s)}{K_2'(s)}$$

for every $s \leq 0$. This inequality shows that Rhode's (2009) more-decreasingly-relatively-impatient-than relation is concerned with both the derivatives and the curvatures of the cumulant-generating function, the two added up with weights being the common coefficient γ of relative risk aversion, and 1, respectively.

6 Comparison within a parametrized family

In many applications, such as those to interest rate models presented in Section , we do not compare two particular distributions μ_1 and μ_2 of individual consumers' discount rates. Rather, we consider a parameterized family of distributions of individual consumers' discount rates, say $(\mu(\cdot, \alpha, \beta))_{(\alpha,\beta)\in\Theta}$ where Θ is an open subset of $\boldsymbol{R}^2$, and determine how the parameter values (α, β) are related to the measure of the representative consumer's decreasing impatience. In this section, we give three such families, consisting of Gamma distributions, Poisson-like distributions, and Bernoulli distributions. We will see that in these classes, only one of the two parameters is crucial for the monotone likelihood ratio property and the single-crossing property. The class of quasi-hyperbolic discount factor functions is outside the scope of our analysis, because they are not continuous at time 0.

6.1 Gamma distributions

First, we consider the family of gamma distributions, each with parameters α and β, that is, its density function (with respect to Lebesgue measure) is given by

$$q \mapsto \frac{\beta^\alpha q^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta q). \tag{16}$$

Gamma distributions are most commonly used as distributions of subjective time discount rates, as in Weitzman (2001), Gollier and Zeckhauser (2005), and Hara (2007).

Define $f(\cdot, \alpha, \beta)$ as the discount factor function when the distribution μ defined in (7) has the density function (16). That is,

$$f(t, \alpha, \beta) = \left(\int_0^\infty \exp\left(-\frac{qt}{\gamma}\right) \frac{\beta^\alpha q^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta q)\, \mathrm{d}q \right)^\gamma.$$

Let $r(\cdot, \alpha, \beta)$ be the discount factor function corresponding to $d(\cdot, \alpha, \beta)$, that is,

$$r(t, \alpha, \beta) = -\frac{\frac{\partial f}{\partial t}(t, \alpha, \beta)}{d(t, \alpha, \beta)}.$$

The more-decreasingly-impatient-than relation can be easily parameterized in the family of Gamma distributions.

Proposition 1 *For all* (α_1, β_1) *and* (α_2, β_2),

$$-\frac{\frac{\partial r}{\partial t}(t, \alpha_1, \beta_1)}{r(t, \alpha_1, \beta_1)} > -\frac{\frac{\partial r}{\partial t}(t, \alpha_2, \beta_2)}{r(t, \alpha_2, \beta_2)}.$$

for every $t \geq 0$ *if and only if* $\beta_1 < \beta_2$.

This proposition states that the parameter α is irrelevant for the ordering of decreasing impatience. Since the family of exponential distributions, which was investigated by Weitzman (2001) and Gollier and Zeckhauser (2005), is a one-parameter subfamily of Gamma distributions parameterized by β (and α fixed at 1), every pair of two distinct exponential distributions of discount rates gives rise to a pair of the representative consumers' discount factor functions that can be strictly ordered by the more-decreasingly-impatient-than relation. On the other hand, the family of chi-squared distributions is a one-parameter subfamily of Gamma distributions parameterized by α (with 2α an integer and α fixed at 2), all chi-squared distributions of discount rates give rise to the representative consumers' discount factor functions that are all equally decreasingly impatient.

The proof of the proposition is simple and goes as follows. The corresponding cumulant-generating function $K(\cdot, \alpha, \beta)$ is given by

$$K(s, \alpha, \beta) = \alpha(\ln \beta - \ln(\beta - s)).$$

Thus

$$\frac{\partial K}{\partial s}(s,\alpha,\beta) = \frac{\alpha}{\beta - s}, \tag{17}$$

$$\frac{\dfrac{\partial^2 K}{\partial s^2}(s,\alpha,\beta)}{\dfrac{\partial K}{\partial s}(s,\alpha,\beta)} = \frac{1}{\beta - s}. \tag{18}$$

Hence, the proposition follows from Theorem 2. The equalities (17) and (18) also show that

$$r(t,\alpha,\beta) = \frac{\alpha\gamma}{t + \beta\gamma}, \tag{19}$$

$$-\frac{\dfrac{\partial r}{\partial t}(t,\alpha,\beta)}{r(t,\alpha_1,\beta_1)} = \frac{1}{t + \beta\gamma} \tag{20}$$

for every t. Thus, the representative consumer has a generalized hyperbolic discount rate function of Loewenstein and Prelec (1992) and the measure of decreasing impatience is, indeed, independent of α; and the role of α is to determine the levels of the representative consumer's discount rates. Moreover, the hyperbolic factor of Rohde (2010) is equal to $(\beta\gamma)^{-1}$, which is constant (an advantage of the hyperbolic factor over our measure of decreasing impatience) and independent of α, and the DI measure of Rohde (2018) coincides with (20).

In concluding this subsection, we note that the Gamma distributions constitute an exponential family,[13] and the the representative consumer's discount rate functions are particularly easy to calculate for an exponential family. To see this, write $\varphi = -\beta$ and $\psi = \alpha - 1$, and define $w(\psi, q) = \psi \ln q$ and $v(\varphi, \psi) = (\psi + 1)\ln(-\varphi) - \ln\Gamma(\psi + 1)$. Then

$$\frac{\beta^\alpha q^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta q) = \exp\left(\varphi q + w(\psi, q) + v(\varphi, \psi)\right). \tag{21}$$

Thus, the Gamma distributions, indeed, constitute an exponential family. With a slight abuse of notation, denote by $K(\cdot, \varphi, \psi)$ the cumulant-generating function that corresponds to parameters (φ, ψ) in the above expression, then $K(s, \varphi, \psi) =$

[13] The definition of a one-dimensional exponential family can be found in Hogg and Craig (1978, Section 4 of Chapter 10). In fact, a family of distributions, parameterized by (φ, ψ) that can be represented by the right-hand side of (21) is more general than an exponential family, because, in (21), $w(\psi, q)$ need not be multiplicatively separable between ψ and q.

$v(\varphi,\psi)-v(\varphi+s,\psi)$. Denote the corresponding discount rate function by $r(\cdot,\varphi,\psi)$. Then, by (9),

$$r(t,\varphi,\psi) = -\frac{\partial v}{\partial \varphi}\left(\varphi - \frac{t}{\gamma}, \psi\right).$$

Thus, for an exponential family, Theorems 2 and 3 can be stated in terms of v.

6.2 Poisson-like distributions

In this subsection, we show that if the distribution of the individual consumers' discount rates is similar to a Poisson distribution, then the representative consumer exhibits constant absolute decreasing impatience in the sense of Bleichrodt, Rohde, and Wakker (2009).

Let $\varepsilon > 0$. We consider the distribution that gives each discount rate $n\varepsilon$ with $n = 0, 1, 2, \dots$ a probability $\exp(-\alpha)\alpha^n/n!$. This is the Poisson distribution with parameter α, except that the probability masses are given to $0, \varepsilon, 2\varepsilon, \dots$ rather than to $0, 1, 2, \dots$. Denote by $K(\cdot,\varepsilon,\alpha)$ its cumulant generating function parameterized by (ε,α). Define $f(\cdot,\varepsilon,\alpha)$ as the discount factor function when the distribution μ defined in (7) coincides with the above distribution. Denote by $r(\cdot,\varepsilon,\alpha)$ the corresponding discount rate function.

Proposition 2 *For every (ε,α) and every $t \geq 0$,*

$$-\frac{\frac{\partial r}{\partial t}(t,\varepsilon,\alpha)}{r(t,\varepsilon,\alpha)} = \frac{\varepsilon}{\gamma}$$

This proposition shows that the representative consumer exhibits constant absolute decreasing impatience in the sense of Bleichrodt, Rohde, and Wakker (2009). The proof follows from $K(s,\varepsilon,\alpha) = \alpha(\exp(s\varepsilon) - 1))$ and

$$r(t,\varepsilon,\alpha) = \frac{\partial K}{\partial s}\left(-\frac{t}{\gamma},\varepsilon,\alpha\right) = \varepsilon\alpha\exp\left(-\frac{\varepsilon}{\gamma}t\right). \tag{22}$$

The proposition also implies that for all (ε_1,α_1) and (ε_2,α_2),

$$-\frac{\frac{\partial r}{\partial t}(t,\varepsilon_1,\alpha_1)}{r(t,\varepsilon_1,\alpha_1)} > -\frac{\frac{\partial r}{\partial t}(t,\varepsilon_2,\alpha_2)}{r(t,\varepsilon_2,\alpha_2)}$$

for every $t \geq 0$ if and only if $\varepsilon_1 > \varepsilon_2$. That is, the more-decreasingly-impatient relation is determined solely by ε. As can be seen in (22), the role of the Poisson parameter α is to determine the levels of the representative consumer's discount rates. Since the utility maximization problem (1) of an individual consumer with the zero discount factor may not have a solution, the fact that these Poisson-like distributions give a positive probability mass $\exp(-\alpha)$ to the zero discount rate is a drawback of these distributions. Yet, according to Bernsteins's theorem and the Inversion Formula,[14] these are the only distributions that give rise to constant absolute decreasing impatience for the representative consumer.

In concluding this subsection, we present another, closely related, use of Bernstein's theorem. Note that the discount rate (not factor) function (22) is a negative exponential function and, thus, a completely monotone function, that is, its first derivative is negative and higher derivatives alternate in sign. This fact can be restated as saying that the derivative of its integral $\int_0^t r(s, \varepsilon, \alpha)\, \mathrm{d}s$ is a completely monotone function of t. Since $f(t, \varepsilon, \alpha) = f(0, \varepsilon, \alpha) \exp\left(-\int_0^t r(s, \varepsilon, \alpha)\, \mathrm{d}s\right)$, we can apply Condition 2 of Feller (1970, Section 4 of Chapter XIII) to show that the corresponding discount factor function $f(\cdot, \varepsilon, \alpha)$ is also a completely monotone function.[15] More generally, whenever the discount rate function r is completely monotone, so is the discount factor functions f. By Bernstein's theorem, therefore, there is a distribution of individual consumers' discount rates with which r is the discount rate function of the representative consumer. In particular, if the discount rate function exhibits relative (not absolute) decreasing impatience in the sense of Bleichrodt, Rohde, and Wakker (2009) with a positive constant, then the discount rate function is a negative power function, which is completely monotone. It is, thus, the representative consumer's discount rate function for some distribution of individual consumers' discount rates.

6.3 Bernoulli distributions

In this subsection, we consider the set of all Bernoulli distributions that put probability $1/2$ to each of $\alpha + \beta$ and $\alpha - \beta$, where $\alpha > \beta > 0$. We define the discount factor function $f(\cdot, \alpha, \beta)$ and the discount rate function $r(\cdot, \alpha, \beta)$ in the same way

[14] These facts are stated as Theorems 1 and 2 of Feller (1970, Section 4 of Chapter XIII). Ebert, Wei, and Zhou (2018) pointed out that constant absolute decreasing impatience can be obtained for a representative consumer's discount factor function, but did not specify the distributions that give rise to constant absolute decreasing impatience.

[15] The use of Condition 2 of Feller (1970, Section 4 of Chapter XIII) in this context is due to Ebert, Wei, and Zhou (2018).

as we did in the previous subsection but using the Bernoulli distributions.

Proposition 3 *For all* (α_1, β_1) *and* (α_2, β_2), $\dfrac{\partial r}{\partial t}(t, \alpha_2, \beta_2) < \dfrac{\partial r}{\partial t}(t, \alpha_1, \beta_1)$ *whenever* $r(t, \alpha_1, \beta_1) = r(t, \alpha_2, \beta_2)$, *if and only if* $\beta_1 > \beta_2$.

Note that the values of the α_n are irrelevant for the ordering of the single-crossing property. The proof is elementary and given in Appendix B.

As for the more-decreasingly-impatient-than relation, note from (45) and (46) that $\partial K(s, \alpha, \beta)/\partial s$ is a strictly increasing function of α but $\partial^2 K(s, \alpha, \beta)/\partial s^2$ does not depend on α. Hence, if $\alpha_1 < \alpha_2$ and $\beta_1 = \beta_2$, $f(\cdot, \alpha_1, \beta_1)$ is more decreasingly impatient than $f(\cdot, \alpha_2, \beta_2)$. We will show that even if $\alpha_1 = \alpha_2$ and $\beta_1 > \beta_2$, $d(\cdot, \alpha_1, \beta_1)$ is not more decreasingly impatient than $d(\cdot, \alpha_2, \beta_2)$. Note that in this case, the Bernoulli distribution with (α_2, β_2) second-order stochastically dominates the Bernoulli distribution (α_1, β_1). This fact shows, thus, that the second-order stochastic dominance relation is not sufficient for the more-decreasingly-impatient-than relation.

Proposition 4 *The partial derivative with respect to* β *of the measure of decreasing impatience,*

$$-\frac{\dfrac{\partial r}{\partial t}(t, \alpha, \beta)}{r(t, \alpha, \beta)}, \tag{23}$$

is positive where t *is sufficiently close to zero and negative where* t *is sufficiently large.*

The proof is elementary and given in Appendix B. This proposition implies that for all α and β with $\alpha > \beta > 0$, a small increase in β does not make the representative consumer more decreasingly impatient or less decreasingly impatient. Rather, as the two possible values, $\alpha + \beta$ and $\alpha - \beta$, of the Bernoulli distribution become further apart, the local measure (23) of decreasing impatience increases in a sufficiently near future, but decreases in a sufficiently distant future. The fact underlying this result was stated, in general terms, in the introduction, but can be more specifically explained as follows. As time goes to infinity, the more patient consumer's share in the average consumption and wealth converges to one, and the less patient consumer's share converges to zero. As can be shown based on the the first-order conditions for the solution to the problem (2), one for the (less patient) consumer with discount rate $\alpha + \beta$, and the other for the (more patient) consumer with discount rate $\alpha - \beta$, the convergence is faster, the larger the value of β is. Thus, if an economy (with a larger β) consists, at the initial time 0, of more heterogeneous individual consumers, then the representative consumer is, for a while, more decreasingly impatient; but as time elapses, the distribution of discount rates, weighted by the wealth shares on intermediate times, becomes less heterogeneous, resulting in a less decreasingly impatient representative consumer.

A numerical example may be of some help to understand this somewhat inconclusive result. Take the mean 4% ($\alpha = 0.04$) and the standard deviation 3% ($\beta = 0.03$).[16] Then the derivative (48) of the curvature (47) of the cumulant-generating function with respect to the standard deviation β turns out to be positive if and only if s is approximately less than 85. According to (14), this implies that if the coefficient of constant relative risk aversion is equal to one (the case of the logarithmic utility function) for both consumers, then a small (infinitesimal) increase in the standard deviation of the binary distribution of individual consumers' discount rates increases the local measure $-r'(t)/r(t)$ of the representative consumer's decreasing impatience during the first 85 years, but such an increase decreases $-r'(t)/r(t)$ thereafter.[17]

In concluding this section, we touch a generalization of the above analysis on the single-crossing property. Note that if Z is a random variable taking values 1 and -1 with probability 1/2 each, then the distribution of its affine transformation $\alpha+\beta Z$ coincides with the Bernoulli distribution with parameter (α, β). Conversely, all Bernoulli distributions are generated by some affine transformations of Z. If, more generally, Z is any random variable of zero mean that is bounded from below, then a family of the distributions of random variables $\alpha + \beta Z$ can be considered as a family of distributions of discount rates. An example of such families is the family of uniform distributions of which the supports are in $\boldsymbol{R}_+$, as in Sozou (1998, Section 4(b)).[18] For such a family, $\partial K(s, \alpha, \beta)/\partial s$ is quasi-linear in α. Hence, $f(\cdot, \alpha_1, \beta_1)$ is more decreasingly impatient than $f(\cdot, \alpha_2, \beta_2)$ whenever $\alpha_1 < \alpha_2$ and $\beta_1 = \beta_2$.

[16]The mean and standard deviation of this numerical example are chosen to match those used by Weitzman (2001, Sections III and IV).

[17]In the context of climate change, 85 years is well within the range of decision making for which a careful choice of discount rates is required.

[18]Take Z so that it follows the uniform distribution, say, on $(-1, 1)$.

7 Applications to interest rate models

In this section, we apply the result in Sections 5 and 6 to interest rate models. In particular, we compare the yield curves and forward rates processes of two exchange economies sharing the same endowment process but having different distributions of individual consumers' discount rates. We conduct a similar comparative exercise for short-rate processes in a separate subsection, because its analysis requires a more careful specification on the average endowment processes. The analysis leads to a broader class of short-rate processes that are tractable and well founded on the equilibrium consideration and, at the same time, accommodate heterogenous impatience. This will be best seen through the hyperbolic extension of the short-rate process of Cox, Ingersoll, and Ross (1985) in the final subsection.

7.1 Yield curves and forward rates

We begin with a close look at the term structure of interest rates of an economy of heterogenous impatience. Assume that all consumers share a common constant coefficient γ of relative risk aversion but have heterogeneous impatience. Denote by μ the distribution of individual consumers' discount rates defined by (7). Then the representative consumer's discount factor function f and discount rate function r are given by (8) and (9). The state-price density process π is equal to the marginal utility process $fu'(e) = (f(t)u'(e(t)))_{t\in\boldsymbol{R}_+}$ evaluated at the average endowment process $e = (e(t))_{t\in\boldsymbol{R}_+}$. Hence the price at time t_1, relative to the current consumption, of the discount bond with maturity $t_2 > t_1$ is equal to

$$E_{t_1}\left(\frac{\pi(t_2)}{\pi(t_1)}\right) = \frac{f(t_2)}{f(t_1)}E_{t_1}\left(\frac{u'(e_{t_2})}{u'(e_{t_1})}\right) = \exp\left(-\int_{t_1}^{t_2} r(t)\,\mathrm{d}t\right)E_{t_1}\left(\frac{u'(e_{t_2})}{u'(e_{t_1})}\right). \quad (24)$$

We denote this price by $B(t_1,t_2)$. The *yield to maturity*, at time t_1, of the discount bond with maturity $t_2 > t_1$ is equal to

$$-\frac{1}{t_1-t_2}\ln B(t_1,t_2) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} r(t)\,\mathrm{d}t - \frac{1}{t_2-t_1}\ln E_{t_1}\left(\frac{u'(e_{t_2})}{u'(e_{t_1})}\right). \quad (25)$$

We denote this $Y(t_1,t_2)$.

Another rate that we are interested in is the *instantaneous forward rate*, determined at time t_1, for the delivery of the about-to-mature bond at time t_2 is equal

to
$$-\frac{\partial}{\partial t_2}\ln B(t_1,t_2) = r(t_2) - \frac{\mathrm{d}}{\mathrm{d}t_2}\ln E_{t_1}\left(\frac{u'(e_{t_2})}{u'(e_{t_1})}\right),$$
if
$$E_{t_1}\left(\frac{u'(e_{t_2})}{u'(e_{t_1})}\right) \tag{26}$$
is a differentiable function of t_2.[19] We denote this by $F(t_1, t_2)$.

In the rest of this subsection, we compare the yield curves and forward rates of two economies having a common average consumption process but different distributions of individual consumers' discount rates. While the literature of the term structure of interest rates has been looking into the conditions under which, say, the yield curve is flat, normal (upward sloping), or abnormal (downward sloping), we will not look into the term structure of each of the two economies in isolation. Rather, we will see how the term structures of the two economies are related to each other as a consequence of the difference in heterogeneity of individual consumers' impatience. We will then determine the term structure of one economy from that of the other via this relation once the latter is known. Lemma 2 in Section A shows that this approach is widely applicable because by assuming that a term structure is obtained at an Arrow-Debreu equilibrium of an economy with heterogeneously impatient consumers imposes no restriction on the term structure.

Let d_1 and d_2 be the discount factor functions derived from two economies, of which the weighting functions are λ_1 and λ_2, the distributions of individual consumers' discount rates are μ_1 and μ_2, just as explained at the beginning of Section 5. That is, for each $n = 1, 2$,
$$\begin{aligned} f_n(t) &= \frac{1}{\left(\int_I \lambda_n^{1/\gamma}\mathrm{d}\iota\right)^\gamma}\left(\int_I (\lambda_n(i)\exp(-\rho_n(i)t))^{1/\gamma}\ \mathrm{d}\iota(i)\right)^\gamma \\ &= \left(\int_0^\infty \exp\left(-\frac{qt}{\gamma}\right)\ \mathrm{d}\mu_n(q)\right)^\gamma. \end{aligned} \tag{27}$$

Denote the cumulant-generating functions of μ_1 and μ_2 by K_1 and K_2. Let r_1 and r_2 be the corresponding discount rate functions. For each $= 1, 2$, let B_n, Y_n, and F_n be the corresponding bond prices, yields to maturity, and instantaneous

[19]This is true if e is an Ito process, as assumed in the next subsection.

forward rates. Then

$$\frac{B_2(t_1,t_2)}{B_1(t_1,t_2)} = \exp\left(-\int_{t_1}^{t_2}(r_1(t)-r_2(t))\,\mathrm{d}t\right),$$

$$Y_1(t_1,t_2) - Y_2(t_1,t_2) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}(r_1(t)-r_2(t))\,\mathrm{d}t, \tag{28}$$

$$F_1(t_1,t_2) - F_2(t_1,t_2) = r_1(t_2) - r_2(t_2), \tag{29}$$

These equalities and Theorem 3 can also be used to establish the single-crossing property of the yield curves and the instantaneous forward rates. The proof of this theorem is given in Appendix B.

Theorem 4 *Suppose that for every $s \leq 0$, $K_1''(s) > K_2''(s)$ whenever $K_1'(s) = K_2'(s)$. Then, for every $t_1 \geq 0$ and every $t_2 > t_1$, if $Y_1(t_1,t_2) = Y_2(t_1,t_2)$, then $Y_1(t_1,t) < Y_2(t_1,t)$ for every $t > t_2$ and $Y_1(t_1,t) > Y_2(t_1,t)$ for every $t \in [t_1,t_2)$; and if $F_1(t_1,t_2) = F_2(t_1,t_2)$, then $F_1(t_1,t) < F_2(t_1,t)$ for every $t > t_2$ and $F_1(t_1,t) > F_2(t_1,t)$ for every $t \in [t_1,t_2)$.*

Theorem 4 compares the instantaneous forward rates, in the two economies, determined at a fixed time t_1 but with a variable delivery time t_2. Another comparison worth exploring is, as in Heath, Jarrow, and Morton (1992), the instantaneous forward rates, with a fixed maturity t_2, but with a variable time t at which the rates are determined. What this means, in symbols, is how

$$F_1(t,t_2) - F_2(t,t_2) \tag{30}$$

depends on $t \leq t_2$. In fact, by (29), (30) is equal to $r_1(t_2) - r_2(t_2)$ independently of t_1. We can conclude, therefore, that the instantaneous forward rates in two economies, with a fixed maturity but with a variable time at which the rates are determined, has a constant difference, rather than the single-crossing property.[20]

7.2 Short-rate processes

In this subsection, we conduct a comparative statics exercise on short-rate processes in the order of decreasing generality. First, to guarantee that the short-rate

[20] The constant difference does depend on the maturity time, and it has the single-crossing property with zero whenever the cumulant-generating functions have the property.

process is well defined, we assume, in addition to the conditions used in the previous subsection, that the average endowment process e is an Ito process. Next, we consider the case in which the short-rate process is itself an Ito process. Finally, we consider the case in which the short-rate process is a solution to a stochastic differential equation. Our aim is twofold. One is to obtain the single-crossing property along the lines of Theorem 4 for short-rate processes. The other is to clarify under what conditions the most commonly used restrictions on short-rate processes, such as the affine structure and the Gaussian structure, can be obtained in the second economy whenever it is assumed for the first economy.

Let the filtration $(\mathscr{F}(t))_{t\in\boldsymbol{R}_+}$ be generated by a one-dimensional standard Brownian motion $B = (B(t))_{t\in\boldsymbol{R}_+}$ and the average endowment process e (where the average is taken over the population I) is a positive-valued Ito process, written as

$$\mathrm{d}e(t) = e(t)\mu_e(t)\,\mathrm{d}t + e(t)\sigma_e(t)\,\mathrm{d}B(t)$$

for some real-valued adapted processes $\mu_e = (\mu_e(t))_{t\in\boldsymbol{R}_+}$ and $\sigma_e = (\sigma_e(t))_{t\in\boldsymbol{R}_+}$. Since $\pi = fu'(c)$, Ito's Lemma implies that π is also a positive-valued Ito process, which can be written as

$$\mathrm{d}\pi(t) = -\pi(t)\eta(t)\,\mathrm{d}t - \pi(t)\kappa(t)\,\mathrm{d}B(t) \tag{31}$$

for some real-valued adapted processes $\eta = (\eta(t))_{t\in\boldsymbol{R}_+}$ and $\kappa = (\kappa(t))_{t\in\boldsymbol{R}_+}$. The process η is the short-rate process and $\eta(t)$ is the instantaneous riskless interest rate at time t. Ito's Lemma also implies that

$$\eta = r + \gamma\mu_e - \frac{\gamma(\gamma+1)}{2}\sigma_e^2. \tag{32}$$

This equality shows how the short-rate process is tied to the average endowment process via μ_e and σ_e, as well as to the representative consumer's discount rate r. It can be used to derive the short-rate process of one economy from that of the other whenever the latter are known.

To compare the short-rate processes of two economies sharing the same average endowment process, suppose that f_1 and f_2 are the discount factor functions derived from two economies as in the previous subsection. The probability measures μ_1 and μ_2, the cumulant-generating functions K_1 and K_2, and the corresponding discount rate functions r_1 and r_2 are defined in the same manner. Denote the short-rate process by η_1 and η_2. By (32),

$$\eta_1 - \eta_2 = r_1 - r_2. \tag{33}$$

That is, the difference in short-rate processes between the two economies is equal to the difference in the representative consumers' discount rates. Although η_1 and η_2 are, in general, stochastic processes, the difference $\eta_1 - \eta_2$ is not, because neither is $r_1 - r_2$. This, along with Theorem 3, is sufficient to establish the following (stochastic) single-crossing property for short-rate processes.

Theorem 5 *Suppose that for every $s \leq 0$, $K_1''(s) > K_2''(s)$ whenever $K_1'(s) = K_2'(s)$. Then, for every $t_1 \geq 0$ if $\eta_1(t_1) = \eta_2(t_1)$, then $\eta_1(t) < \eta_2(t)$ for every $t > t_1$ and $\eta_1(t) > \eta_2(t)$ for every $t < t_1$.*

Since $\eta_1 - \eta_2$ is a process that is not stochastic, the time at which the two short-rate processes η_1 and η_2 take the same value is not stochastic either. The above theorem, thus, implies that there is a deterministic time before which the short rate is higher in the first economy than in the second; and after which it is lower in the first economy than in the second.

The relation (33) also allows us to show that if one of the two has an affine structure, so does the other, and how the two are related to each other.

Theorem 6 *Suppose that there are two functions $h_1 : \boldsymbol{R}_+^2 \to \boldsymbol{R}$ and $h : \boldsymbol{R}_+^2 \to \boldsymbol{R}$ such that*

$$Y_1(t_1, t_2) = h_1(t_1, t_2) + h(t_1, t_2)\eta_1(t_1)$$

for every (t_1, t_2) with $t_1 < t_2$. Define a function $h_2 : \boldsymbol{R}_+^2 \to \boldsymbol{R}$ by

$$h_2(t_1, t_2) = h_1(t_1, t_2) - \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (r_1(t) - r_2(t))\, \mathrm{d}t,$$

then

$$Y_2(t_1, t_2) = h_2(t_1, t_2) + h(t_1, t_2)\eta_2(t_1).$$

This theorem shows, in particular, that the coefficients to the short rate, $h(t_1, t_2)$, are equal between the two economies. It follows immediately from (28) and (33).

The other process κ that appears in the state price process (31) is known as the market-price-of-risk process, as it represents the adjustment necessary to transform the original probability measure to the equivalent martingale measure. Again by

Ito's Lemma, $\kappa = \gamma\sigma_e$. Thus, it is independent of the representative consumer's impatience. This independence, however, has an important implication on the applicability of our results, which we explain in the next paragraph.

Suppose in addition that the short-rate process η is an Ito process that can be written as

$$\mathrm{d}\eta(t) = \mu_\eta(t)\,\mathrm{d}t + \sigma_\eta(t)\,\mathrm{d}B(t) \tag{34}$$

for some real-valued adapted processes $\mu_\eta = (\mu_\eta(t))_{t\in\boldsymbol{R}_+}$ and $\sigma_\eta = (\sigma_\eta(t))_{t\in\boldsymbol{R}_+}$. This is an expression of the short-rate process in terms of the physical (natural) measure P on the state space Ω, as B is a standard Brownian motion under P. In the literature on short-rate processes, however, it is customary to represent them in terms of a standard Brownian motion under the equivalent martingale measure. The convenience of doing so can be seen in the price (25) of the discount bond. Using the equivalent martingale measure Q, we can rewrite the bond price as

$$E^Q_{t_1}\left(\exp\left(-\int_{t_1}^{t_2}\eta(t)\,\mathrm{d}t\right)\right)$$

and, to calculate this expectation with respect to Q, in the case of Gaussian models, for example, it would be better if η is written as an Ito process in terms of a standard Brownian motion under Q. It can be derived from the market-price-of-risk process κ by defining another Ito process υ by $\upsilon(0) = 0$ and $\mathrm{d}\upsilon(t) = -\kappa(t)\upsilon(t)\mathrm{d}B(t)$, and, then, by letting

$$E_t\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right) = \upsilon(t)$$

for every t. Then define another Ito process B^Q by $B^Q(0) = 0$ and $\mathrm{d}B^Q(t) = \mathrm{d}B(t) + \kappa(t)\mathrm{d}t$. By Girsanov's theorem, B^Q is a standard Brownian motion under Q. Then the short-rate process (34) can be written in terms of the Brownian motion B^Q with respect to Q as

$$\mathrm{d}\eta(t) = (\mu_\eta(t) - \sigma_\eta(t)\kappa(t))\,\mathrm{d}t + \sigma_\eta(t)\,\mathrm{d}B^Q(t).$$

This is an expression of the short-rate process under the equivalent martingale measure Q. We have seen that the market-price-of-risk process κ does not depend on the heterogeneity of the individual consumers' discount rates. We will also see that σ_η does not depend on the heterogeneity either. Only μ_η depends on the heterogeneity. Thus, all results on the diffusion term of the short-rate process, and all results on the difference in the drift term between two economies, that are valid under the physical measure P are also valid under the equivalent martingale measure Q. It is, therefore, sufficient to state all results under the physical measure P.

The following theorem gives the relationship between the drift and diffusion terms of the two short-rate processes. Specifically, it shows that the difference in the drift terms is equal to the difference in the reductions of the representative consumers' discount rates, and that the diffusion terms are the same. If follows immediately from (33).

Theorem 7 *Suppose that the short-rate process η_1 of the first economy is an Ito process written as*

$$\mathrm{d}\eta_1(t) = \mu_1(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t) \tag{35}$$

for some $\boldsymbol{R}$-valued adapted processes $\mu_1 = (\mu_1(t))_{t\in\boldsymbol{R}_+}$ and $\sigma = (\sigma(t))_{t\in\boldsymbol{R}_+}$. Define another adapted processes μ_2 by $\mu_2 = \mu_1 - (r_1' - r_2')$. Then the short-rate process η_2 of the second economy is also an Ito process, written as

$$\mathrm{d}\eta_2(t) = \mu_2(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t). \tag{36}$$

Note that in Theorem 7, the short-rate process η_n is represented via a stochastic integration, not as the solution to a stochastic differential equation. The latter is predominantly used in the literature on short-rate processes, because, then, the conditional distribution of short rates depends only on the current short rate, not on the past ones. The class of affine structure models is such an example. The following corollary of Theorem 7 deals specifically with the short-rate processes that are solutions to stochastic differential equations. The proof is given in Appendix B.

Corollary 1 1. *Suppose that the short rate process η_1 of the second economy is the solution to a stochastic differential equation*

$$\mathrm{d}\eta_1(t) = h_1(\eta_1(t), t)\,\mathrm{d}t + g_1(\eta_1(t), t)\,\mathrm{d}B(t). \tag{37}$$

for some functions $h_1 : \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}$ and $g_1 : \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}$. Then, *define two functions* $h_2 : \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}$ *and* $g_2 : \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}$ *by*

$$h_2(x,t) = h_1(x + (r_1(t) - r_2(t)), t) - (r_1'(t) - r_2'(t)), \tag{38}$$

$$g_2(x,t) = g_1(x + (r_1(t) - r_2(t)), t). \tag{39}$$

Then the short-rate process η_2 *of the second economy is the solution to the stochastic differential equation*

$$\mathrm{d}\eta_2(t) = h_2(\eta_2(t), t)\,\mathrm{d}t + g_2(\eta_2(t), t)\,\mathrm{d}B(t) \tag{40}$$

2. *Suppose that the short rate process* η_1 *of the first economy is the solution to a stochastic differential equation*

$$\mathrm{d}\eta_1(t) = (h_1(t) + h(t)\eta_1(t))\,\mathrm{d}t + (g_1(t) + g(t)\eta_1(t))^k\,\mathrm{d}B(t). \tag{41}$$

for some functions $h_1 : \boldsymbol{R}_+ \to \boldsymbol{R}$*,* $h : \boldsymbol{R}_+ \to \boldsymbol{R}$*,* $g_1 : \boldsymbol{R}_+ \to \boldsymbol{R}$*,* $g : \boldsymbol{R}_+ \to \boldsymbol{R}$*, and a* $k \geq 1/2$*. Define two functions* $h_2 : \boldsymbol{R}_+ \to \boldsymbol{R}$ *and* $g_2 : \boldsymbol{R}_+ \to \boldsymbol{R}$ *by*

$$h_2(t) = h_1(t) + h(t)(r_1(t) - r_2(t)) - (r_1'(t) - r_2'(t)),$$

$$g_2(t) = g_1(t) + g(t)(r_1(t) - r_2(t)),$$

then the short rate process η_2 *of the second economy is the solution to the stochastic differential equation*

$$\mathrm{d}\eta_2(t) = (h_2(t) + h(t)\eta_2(t))\,\mathrm{d}t + (g_2(t) + g(t)\eta_2(t))^k\,\mathrm{d}B(t).$$

The first part of this corollary shows that if the short-rate process of the first economy is given as the solution to a stochastic differential equation, then so is the short-rate process of the second economy, and show also how the functions defining the latter can be obtained from those of the former. The second part shows that if the short-rate process of the first economy has the affine structure, then so does the short-rate process of the second economy, and show also how the functions defining the latter can be obtained from those of the former. In particular, it implies that if η_1 is Gaussian, then $k = 1$ and g is constantly equal to zero, and, thus, η_2 is also Gaussian; and that η_1 is a short-rate process of Ho and Lee, then the three functions f, g_1, and g are all constant, and the first and last of these three always take value zero, and, thus, η_2 is also a short-rate process of Ho and Lee.

7.3 Hyperbolic CIR model

As an application of Corollary 1, we present, in this subsection, how the the term structure of interest rates of Cox, Ingersoll, and Ross (1985) is affected when the representative consumer has a generalized hyperbolic discounting rate function of Lowenstein and Prelec (1992). This case is interesting because the (generalized) hyperbolic discounting is a most commonly used discounting that exhibits decreasing impatience, and also because the form of the resulting short-rate process turns out to be different from the CIR form. That the process is given in such an explicit form deserves special attention, as deriving such an explicit form of yield curves and short-rate processes in economies of heterogeneous consumers have been known to be a formidable task, as can be seen in Dumas (1989) and Wang (1996).

We take the first economy as an economy of the CIR model with the common logarithmic utility function ($\gamma = 1$), a common discount rate $\bar{\rho}$, and a common endowment process. Formally, we take $I = (0, 1)$, $\mathscr{I}$ to be the set of all Lebesgue measurable subsets of I, and ι be the Lebesgue measure restricted on I. Both $\rho_1 : I \to \boldsymbol{R}_{++}$ and $\theta_1 : I \to \boldsymbol{R}_{++}$ are constant, and the former always takes value $\bar{\rho}$ and the latter always takes value 1. Let η_1 a short-rate process of the CIR model, that is, suppose that there are a $k_0 \in \boldsymbol{R}_{++}$, a $k_1 \in \boldsymbol{R}_{++}$, and an $\bar{\eta} \in \boldsymbol{R}_{++}$ such that

$$\mathrm{d}\eta_1(t) = k_0(\bar{\eta} - \eta_1(t))\,\mathrm{d}t + k_1\sqrt{\eta_1(t)}\,\mathrm{d}B(t). \tag{42}$$

Let κ be any positive-valued process and define the average endowment process e, the utility weights ρ_1, the wealth shares θ_1, and the state-price density process π_1 with $\mathrm{d}\pi_1(t) = -\eta_1(t)\pi_1(t)\,\mathrm{d}t - \kappa(t)\pi_1(t)\,\mathrm{d}B(t)$ via Lemma 2.[21]

Then, take the second economy as an economy in which all the individual consumers have the logarithmic utility and the same endowment process e, but their discount rates are distributed according to a Gamma distribution of which the mean is equal to $\bar{\rho}$, the common discount factor of the first economy. The following proposition, proved in Appendix B, gives the short-rate process η_2 of the

[21]The average endowment process e coincides with the equilibrium consumption process of the representative agent of the CIR model if we let $\kappa = \sqrt{\eta_1}$.

second economy.

Proposition 5 *If* $\theta_2 : I \to \boldsymbol{R}_{++}$ *constantly takes value* 1 *and* $\rho_2 : I \to \boldsymbol{R}_{++}$ *coincides with the inverse of the cumulative distribution function of the Gamma distribution* (16) *satisfying* $\bar{\rho} = \alpha/\beta$, *then*

$$\begin{aligned} \mathrm{d}\eta_2(t) = \quad & k_0 \left(\left(\bar{\eta} - \left(\frac{t}{t+\beta} - \frac{\beta}{k_0(t+\beta)^2} \right) \bar{\rho} \right) - \eta_2(t) \right) \mathrm{d}t \\ & + k_1 \sqrt{\eta_2(t) + \frac{t}{t+\beta}\bar{\rho}}\, \mathrm{d}B(t). \end{aligned} \tag{43}$$

The inverse of any continuous and strictly increasing cumulative distribution function is a (continuous and strictly increasing) function on $(0,1)$ and its cumulative distribution function, with respect to the Lebesgue measure on $(0,1)$, coincides with the given cumulative distribution function.[22] Thus, in Proposition 5, together with the assumption of equal wealth shares, the distribution of the individual consumers' discount rates coincides with the Gamma distribution with parameters (α, β). The equality $\bar{\rho} = \alpha/\beta$ says that the parameters (α, β) are chosen so that the mean of the individual consumers' discount rates of the second economy is equal to $\bar{\rho}$. The parameter β, as proved in Proposition 1, measures the heterogeneity of individual consumers' impatience and, thus, the decreasing impatience of the representative consumer's discount factor function, with the smaller β leading to the more heterogenous individual consumers and the more decreasingly impatient representative consumer. Thus, (43) gives a family of hyperbolic extensions of the CIR model, where the mean of the individual consumers' discount rates is fixed at $\bar{\rho}$ and the representative consumer's decreasing impatience is parameterized by β.

The short-rate process η_2 of the heterogeneous economy has the same speed of adjustment, k_0 as the short-rate process η_1 of the homogeneous economy, but has a different mean of reversion. Indeed, the mean of reversion is

$$\bar{\eta} - \left(\frac{t}{t+\beta} - \frac{\beta}{k_0(t+\beta)^2} \right) \bar{\rho},$$

which is deterministic but time-varying, decreasing strictly from $\bar{\eta} + \bar{\rho}/k_0\beta$ to $\bar{\eta} - \bar{\rho}$ as $t \to \infty$. Keeping t fixed but varying β instead, we see that it increases strictly

[22] This is a special case of Proposition 3.1(2) of Embrechts and Hofert (2014) on the generalized inverses of cumulative distribution functions, which need not be continuous or strictly increasing.

from $\bar{\eta}-\bar{\rho}$ to $\bar{\eta}$ as $\beta \to \infty$. Since the larger β leads to the less heterogenous individual consumers and the more decreasingly impatient representative consumer, this means that as the individual consumers' impatience becomes more heterogenous, and hence the representative consumer becomes more decreasingly impatient, the mean of reversion decreases, from the homogeneous benchmark level $\bar{\rho}$ to $\bar{\eta} - \bar{\rho}$.

8 Conclusion

In this paper, we have given a precise formulation to the notion that the more heterogeneous the individual consumers' subjective discount rates are, the more decreasingly impatient the representative consumer is. The measure of heterogeneity of discount rates is the convexity of the cumulant-generating function of the (approximately wealth-weighted) distribution of discount rates. We have also given two examples of parameterized families of distributions within which the measures of heterogeneity are compared, of which one consists of Gamma distributions and the other consists of Bernoulli distributions. We have applied these results to the analysis of the term structure of interest rates in heterogeneous economies. In particular, we have characterized the short-rate process in the version of the CIR in a heterogeneous economy.

In the future research, the analysis of this paper should be extended to the case where asset markets are incomplete. Since individual consumers have fewer instruments to transfer purchasing power across time and states, the impact of the heterogeneity of individual consumers' impatience on the representative consumer's impatience will be less pronounced than in the case of complete markets. To increase the relevance of the results of this paper, it is important to determine exactly how much the impact is reduced.

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A Equilibrium foundation of our comparative statics exercises

In this appendix, we results two lemmas that are useful when assessing the changes in the degree of heterogeneity of individual consumers' impatience and in the degree of the representative consumer's decreasing impatience that are caused by changes in individual consumers' impatience or wealth levels (or bot).

Under an Arrow-Debreu equilibrium state-price density process π, define the wealth share function $\theta : I \to \boldsymbol{R}_{++}$ by

$$\theta(i) = \frac{E\left(\int_0^\infty \pi(t) e_i(t)\,\mathrm{d}t\right)}{E\left(\int_0^\infty \pi(t) e(t)\,\mathrm{d}t\right)}, \tag{44}$$

then $\int_I \theta(i)\mathrm{d}\iota(i) = 1$. The following lemma relates the utility weights λ to the wealth shares θ when all consumers have the logarithmic utility function. Since the proof is straightforward, we omit it.

Lemma 1 *Suppose that $\gamma = 1$. Let $\lambda : I \to \boldsymbol{R}_{++}$. If the solution to the social welfare maximization problem* (2) *coincides with the equilibrium allocation when $c = e$, then λ is a scalar multiple of $\theta\rho$.*

This lemma can be used to conduct a comparative statics exercise on the representative consumers' discount factor functions in the following manner. First, we arbitrarily fix the average endowment process e and the space $(A, \mathscr{I}, \iota)$ of consumers. Assume that all consumers have the logarithmic utility, that is, $\gamma = 1$. Then, define the first economy by letting consumer i have the discount rate $\rho_1(i)$ and the endowment process $\theta_1(i)e$. This defines two functions $\rho_1 : I \to \boldsymbol{R}_{++}$ and $\theta_1 : I \to \boldsymbol{R}_{++}$. Since the individual consumers' endowment processes are all scalar multiples of the average endowment process, the wealth shares are given by the function θ_1, regardless of the equilibrium state-price deflator π. Thus the representative consumer's discount factor function f_1 of the first economy is given by

$$f_1(t) = \int_I \lambda_1(i) \exp(-\rho_1(i)t)\,\mathrm{d}\iota(i),$$

where $\lambda_1 = \left(\int_I \theta_1\rho_1\,\mathrm{d}\iota\right)^{-1} \theta_1\rho_1 : I \to \boldsymbol{R}_{++}$. Equivalently, if we define the probability measure μ_1 on $\mathscr{B}(\boldsymbol{R}_{++})$ in the same way as in Section 3 using ρ_1 and θ_1, then

$$f_1(t) = \int_{\boldsymbol{R}_{++}} \exp(-qt)\,\mathrm{d}\mu_1(q).$$

If the discount rates and the endowment process of consumer i are changed into $\rho_2(i)$ and $\theta_2(i)e$ (while keeping the population $(A, \mathscr{I}, \iota)$ and the average endowment process e fixed and maintaining the logarithmic utility function), then we can define $\lambda_2 : I \to \boldsymbol{R}_{++}$ and μ_2 in the same manner but using ρ_2 and θ_2 in place of ρ_1 and θ_1 to obtain the representative consumer's discount factor function after the change:

$$f_2(t) = \int_I \lambda_2(i) \exp(-\rho_2(i)t)\,\mathrm{d}\iota(i) = \int_{\boldsymbol{R}_{++}} \exp(-qt)\,\mathrm{d}\mu_2(q).$$

Thus, it is relatively easy to keep track of the change in the individual consumers' discount rates and wealth shares that induces, at equilibrium, a change in the representative consumer's discount factor function, from f_1 to f_2.

To gauge the applicability of this approach, it is important to know what kind of restrictions are imposed on the state-price density process by assuming that it is derived from aggregating the individual consumers' heterogeneous impatience. The following lemma shows that even when the individual consumers' discount rates and wealth shares are arbitrarily fixed and they are all assumed to have the logarithmic utility function, the derivation does not impose any restriction on the state-price density process, aside from its positive-valuedness, as long as the average endowment process can be appropriately chosen.

Lemma 2 *Let $\rho : I \to \boldsymbol{R}_{++}$ and $\theta : I \to \boldsymbol{R}_{++}$. Assume that $\int_I \theta(i)\,\mathrm{d}\iota(i) = 1$. Let π be a positive-valued adapted process. Then, there is an average endowment process e such that π is the state-price density process at the Arrow-Debreu equilibrium of the exchange economy in which every consumer $i \in I$ has the logarithmic utility function, the discount rate $\rho(i)$, and the endowment process $\theta(i)e$.*

Since $\theta(i)e$ is the endowment process of each consumer $i \in I$, all individual consumers' endowment processes are scalar multiples of the average endowment process. The wealth shares evaluated at any equilibrium of this economy coincides with θ.

Proof of Lemma 2 Define a discount factor function f by (4) where $\gamma = 1$ and

$\lambda = \left(\int_I \theta\rho \, \mathrm{d}\iota\right)^{-1} \theta\rho$. Define $e = f/\pi$. We prove that e has the properties stated in the lemma.

For each $i \in I$, define the endowment process $e_i = \theta(i)e$ and a consumption process c_i by

$$c_i(t) = \frac{\lambda(i)\exp(-\rho(i)t)}{f(t)} e(t)$$

Then $\int_I c_i(t) \, \mathrm{d}\iota(i) = e(t)$. Thus the resource-feasibility condition is met. By $\pi e = d$,

$$E\left(\int_0^\infty \pi(t)c_i(t) \, \mathrm{d}t\right) = \frac{\theta(i)}{\int_I \theta\rho \, \mathrm{d}\iota} \int_0^\infty \rho(i)\exp(-\rho(i)t) \, \mathrm{d}t = \frac{\theta(i)}{\int_I \theta\rho \, \mathrm{d}\iota}.$$

By Fubini's theorem,

$$\begin{aligned} E\left(\int_0^\infty \pi(t)e(t) \, \mathrm{d}t\right) &= \int_0^\infty f(t) \, \mathrm{d}t \\ &= \frac{1}{\int_I \theta\rho \, \mathrm{d}\iota} \int_I \theta(i) \left(\int_0^\infty \rho(i)\exp(-\rho(i)t) \, \mathrm{d}t\right) \mathrm{d}\iota(i) \\ &= \frac{1}{\int_I \theta\rho \, \mathrm{d}\iota} \int_I \theta(i) \mathrm{d}\iota(i) = \frac{1}{\int_I \theta\rho \, \mathrm{d}\iota}. \end{aligned}$$

Thus, by the definition of e_i, the budget constraint is met. Moreover,

$$\begin{aligned} &\frac{1}{\pi(t)} \exp(-\rho(i)t) \frac{1}{c_i(t)} \\ =&\frac{1}{\pi(t)} \exp(-\rho(i)t) \frac{f(t)}{\lambda(i)\exp(-\rho(i)t)} \frac{1}{e(t)} = \frac{1}{\lambda(i)}, \end{aligned}$$

which is deterministic and independent of t. Since the utility function is logarithmic, this implies that the utility maximization condition is met. Hence the state-price density process π and the consumption processes $(c_i)_{i \in I}$ constitute an Arrow-Debreu equilibrium. ///

B Proofs

Proof of Proposition 3 The cumulant-generating function $K(\cdot,\alpha,\beta)$ is given by

$$K(s,\alpha,\beta) = \ln\left(\frac{1}{2}\exp\left((\alpha+\beta)s\right) + \frac{1}{2}\exp\left((\alpha-\beta)s\right)\right).$$

Hence,

$$\begin{aligned}
\frac{\partial K}{\partial s}(s,\alpha,\beta) &= \alpha + \beta\frac{\exp(\beta s) - \exp(-\beta s)}{\exp(\beta s) + \exp(-\beta s)}, && (45)\\
\frac{\partial^2 K}{\partial s^2}(s,\alpha,\beta) &= \frac{\beta^2}{(\exp(\beta s) + \exp(-\beta s))^2} \\
&\quad \times \left((\exp(\beta s) + \exp(-\beta s))^2 - (\exp(\beta s) - \exp(-\beta s))^2 \right) \\
&= \left(\frac{2\beta}{\exp(\beta s) + \exp(-\beta s)}\right)^2. && (46)
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{\beta}{\exp(\beta s) + \exp(-\beta s)}\right) \\
&= (\exp(\beta s) + \exp(-\beta s))^{-1} + \beta^2 (\exp(\beta s) + \exp(-\beta s))^{-2} (\exp(-\beta s) - \exp(\beta s)) > 0
\end{aligned}$$

for every $s \leq 0$, (46) is a strictly increasing function of β. Thus the single-crossing property between the $K(\cdot,\alpha_1,\beta_1)$ and $K(\cdot,\alpha_2,\beta_3)$ and, hence, the single-crossing property between $r(\cdot,\alpha_1,\beta_1)$ and $r(\cdot,\alpha_2,\beta_2)$ stipulated in Theorem 3 hold if $\beta_1 > \beta_2$.

At any $s < 0$, the second term on the right-hand side of (45) is a strictly decreasing function of β. Thus, if $(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) \leq 0$, then $\partial K(\cdot,\alpha_1,\beta_1)/\partial s$ and $\partial K(\cdot,\alpha_2,\beta_2)/\partial s$ never intersect. Hence, if they do in fact intersect and satisfy the single-crossing property of Theorem 3, then $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$, that is, the converse of the above claim holds. ///

Proof of Proposition 4 By (45) and (46),

$$\frac{\dfrac{\partial^2 K}{\partial s^2}(s,\alpha,\beta)}{\dfrac{\partial K}{\partial s}(s,\alpha,\beta)} = 4\beta^2\left((\alpha+\beta)\exp(\beta s)+(\alpha-\beta)\exp(-\beta s)\right)^{-1}\left(\exp(\beta s)+\exp(-\beta s)\right)^{-1}$$
$$= 4\beta^2\left((\alpha+\beta)\exp(2\beta s)+(\alpha-\beta)\exp(-2\beta s)+2\alpha\right)^{-1}. \tag{47}$$

Differentiate the curvature (47) with respect to β, then we obtain

$$\begin{aligned}
&\left((\alpha+\beta)\exp(2\beta s)+(\alpha-\beta)\exp(-2\beta s)+2\alpha\right)^{-2}\\
&\times\Big(8\beta\left((\alpha+\beta)\exp(2\beta s)+(\alpha-\beta)\exp(-2\beta s)+2\alpha\right)\\
&\qquad-4\beta^2\left((1+2(\alpha+\beta)s)\exp(2\beta s)-(1+2(\alpha-\beta)s)\exp(-2\beta s)\right)\Big)\\
=\,&4\beta\left((\alpha+\beta)\exp(2\beta s)+(\alpha-\beta)\exp(-2\beta s)+2\alpha\right)^{-2}\\
&\times\left(((2\alpha+\beta)-2(\alpha+\beta)\beta s)\exp(2\beta s)+((2\alpha-\beta)+2(\alpha-\beta)\beta s)\exp(-2\beta s)+4\alpha\right).
\end{aligned} \tag{48}$$

We claim that (48) is strictly positive for every $s \le 0$ sufficiently close to 0 but strictly negative for every sufficiently negative s. First, if $s=0$, then (48) is equal to $2\beta/\alpha$, which is strictly positive. Thus, (48) is strictly positive for every $s \le 0$ sufficiently close to 0. As for a sufficiently negative s, since

$$4\beta\left((\alpha+\beta)\exp(2\beta s)+(\alpha-\beta)\exp(-2\beta s)+2\alpha\right)^{-2} > 0$$

for every s, it suffices to show that

$$((2\alpha+\beta)-2(\alpha+\beta)\beta s)\exp(2\beta s)+((2\alpha-\beta)+2(\alpha-\beta)\beta s)\exp(-2\beta s)+4\alpha < 0 \tag{49}$$

for every sufficiently large s. To do so, note that the first term of the left-hand side of (49), $((2\alpha+\beta)-2(\alpha+\beta)\beta s)\exp(2\beta s)$, converges to zero as $s\to-\infty$. The second term, $((2\alpha-\beta)+2(\alpha-\beta)\beta s)\exp(-2\beta s)$, diverges to $-\infty$ as $s\to-\infty$. The third term, 4α, does not depend on s. Thus, the left-hand side of (49) diverges to $-\infty$ as $s\to-\infty$, and, therefore, is strictly negative for every sufficiently negative s. The proposition then follows from (14). ///

Proof of Theorem 4 The single-crossing property of instantaneous forward rates follows from Theorem 3 and (29). As for yields to maturity, by (28) and a straightforward calculation,

$$\frac{\partial Y_1}{\partial t_2}(t_1,t_2) - \frac{\partial Y_1}{\partial t_2}(t_1,t_2) = \frac{1}{t_2-t_1}\left((r_1(t_2)-r_2(t_2)) - (Y_1(t_1,t_2)-Y_2(t_1,t_2))\right).$$

Thus, if $Y_1(t_1,t_2) = Y_2(t_1,t_2)$, then

$$\frac{\partial Y_1}{\partial t_2}(t_1,t_2) - \frac{\partial Y_1}{\partial t_2}(t_1,t_2) = \frac{1}{t_2-t_1}\left(r_1(t_2)-r_2(t_2)\right), \tag{50}$$

and, again by (28),

$$\int_{t_1}^{t_2} (r_1(t) - r_2(t))\, \mathrm{d}t = 0.$$

Thus, there exists a $t_0 \in (t_1, t_2)$ such that $r_1(t_0) = r_2(t_0)$. By the single-crossing property of the K_n' and Theorem 2, the r_n also have the single-crossing property. Since $t_2 > t_0$, $r_1(t_2) < r_2(t_2)$. By (50),

$$\frac{\partial Y_1}{\partial t_2}(t_1,t_2) - \frac{\partial Y_1}{\partial t_2}(t_1,t_2) < 0.$$

///

Proof of Corollary 1

1. (40) follows from (33) and Theorem 7.
2. This part can be obtained from the first part by taking $h_1(x,t)$ and $g_1(x,t)$ in the first part to be $h_1(t) + h(t)x$ and $(g_1(t) + g(t)x)^k$ in the statement of this part.

///

Proof of Proposition 5 The short-rate process (42) can be written in the form of (41) by letting $k = 1/2$ and

$$\begin{aligned} h_1(t) &= k_0\bar{\eta}, \\ h(t) &= -k_0, \\ g_1(t) &= 0, \\ g(t) &= k_1^2. \end{aligned}$$

As stated right after the statement of Proposition 5, the (wealth-weighted) distribution of the individual consumers' discount rates in the first economy coincides with the Gamma distribution with parameters (α, β) with $\bar{\rho} = \alpha/\beta$. Since $\gamma = 1$ and $\alpha = \bar{\rho}\beta$,

$$
\begin{aligned}
r_1(t) - r_2(t) &= -\frac{\bar{\rho}\beta}{t+\beta} + \bar{\rho} = \frac{\bar{\rho}t}{t+\beta}, \\
r_1'(t) - r_2'(t) &= \frac{\bar{\rho}\beta}{(t+\beta)^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
h_1(t) + h(t)(r_1(t) - r_2(t)) - (r_1'(t) - r_2'(t)) &= k_0\bar{\eta} - k_0\frac{\bar{\rho}t}{t+\beta} - \frac{\bar{\rho}\beta}{(t+\beta)^2}, \\
&= k_0\left(\bar{\eta} - \bar{\rho}\left(\frac{t}{t+\beta} - \frac{\beta}{k_0(t+\beta)^2}\right)\right), \\
g_1(t) + g(t)(r_1(t) - r_2(t)) &= k_1^2\frac{\bar{\rho}t}{t+\beta},
\end{aligned}
$$

Thus, (43) follows from part 2 of Corollary 1. ///