General Equilibrium with Incomplete Financial Markets: Introduction

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Chapter 1

Introduction and Purpose

The general equilibrium theory is an idealized analytical tool for perfectly competitive markets. By perfect competition we mean that all the economic agents adopt “price taking behavior” by assumption, which would occur in an idealized environment where agents are too small to affect markets’ outcomes. The theory describes the markets for all the goods in question. A competitive equilibrium explains all the market prices as well as the resource allocation endogenously: prices are determined simultaneously in such a way that the demand and the supply are equalized in all the markets.

Why do we want to consider all the markets (as well as all potential consumption possibilities)? This is because we do not get a fully satisfactory answer about the determination of prices as well as the associated resource allocation in the so called the partial equilibrium analysis, which looks at a specific market in question taking other prices and income distribution etc. as fixed. Economic variables are affected each other, and so the partial equilibrium analysis does not explain:

• how changes in economic characteristics concerning one good influences the whole economy,

• how income/wealth is generated and distributed to the economic agents,

• how efficiently the scarce resources are used in the whole economy.

Hence the economic intuition built upon the partial equilibrium analysis, although they are extremely useful, fails to capture these issues and so it is sometimes misleading. It is thus only after we learn the general equilibrium theory that we understand the complete flow of economic resources and how they are utilized efficiently/inefficiently.

A fundamental question in the general equilibrium theory is whether or not competitive markets work; that is, one wants to ask if the markets distribute resources in a good manner. And a short answer to this turns out to be yes, as far as efficiency is concerned, but with an important assumption: markets have to be complete in the sense that all the relevant goods are traded in the markets. If some relevant markets are missing, then markets might not perform well.

The purpose of this lecture notes is to provide an introduction to the general equilibrium theory with missing markets. In particular we shall focus on financial
markets, i.e., some markets are missing because there is only insufficient financial instruments - general equilibrium with incomplete financial markets (GEI).\footnote{A standard textbook for this topic is Magill and Quinzii (1992). For the topics about the complete markets, Mas-Colell (1985) is a basic reference. Balasko (1988) is also a useful book on the differentiable technique.}
Chapter 2

Leading example: an Equilibrium model for interest rate with background risks.

I shall describe a simple model of competitive markets with background risks.

2.1 Set up

Consider a two period economy with one perishable good in each period, 0 and 1. There are $I$ consumers. Consumer $i$’s preferences are represented by a concave, differentiable (time additive) von Neumann Morgenstern (vNM) utility function $u_i$, $i = 1, ..., I$, with discount factor $\delta \in (0, 1)$. Consumer $i$ is endowed with $e^0_i$ units of the good in period 0, and his endowment of the good in period 1 is random, and it is represented by a random variable $Y_i$.

There is a market for a riskless discount bond, which is a security which promises to pay one unit of good in the second period for sure. So the bond is an inflation indexed bond. Assume that the net supply of the bond is $B$. Denote by $b_i$ the amount of the discount bond the consumer $i$ chooses to own (note: $b_i$ may be negative - in that case agent $i$ is lending). The price of the bond is $q$.

Thus in the first period, consumer $i$’s choice of consumption $x^0_i$ is subject to a budget constraint:

$$x^0_i + qb_i = e^0_i.$$  

And in the second period, since each consumer can do nothing but the whole income, consumer $i$’s choice of (random) consumption $X_i$ is subject to a budget constraint:

$$X_i = b_i + Y_i.$$  

Or if we write $S$ the underlying state space for the random variables, the equation above can be read as

$$x^s_i = b_i + Y_i(s) \text{ for all } s \in S.$$  

The utility from a consumption plan $(x^0_i, X_i)$ is

$$u_i(x^0_i) + \delta_i E[u_i(X_i)]$$

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and consumer $i$ will choose a consumption plan which maximizes this objective function given the budget constraints above.$^1$

One can solve the budget constraints to eliminate consumption $x_i$ in the problem described above. Consequently, consumer $i$ solves, taking prices as given, the following problem:

$$\max_{b_i} u_i \left(e^0 - qb_i \right) + \delta_i E \left[u_i \left(b_i + Y_i \right)\right] \tag{2.1}$$

where $E$ is the expectation operator. If we denote the underlying (finite) state space as $S$, the expression above can of course be written as

$$\max_{x_i} u_i \left(e^0 - qb_i \right) + \delta_i \sum_{s \in S} \pi(s) \left[u_i \left(b_i + Y_i(s) \right)\right]$$

From the FOC for the consumer’s problem, we have:

$$-qu_i' \left(e^0 - qb_i \right) + \delta_i E \left[u_i' \left(b_i + Y_i \right)\right] = 0. \tag{2.2}$$

The left hand side of the equation above is decreasing in $b_i$ since $u''_i < 0$ (risk aversion) by assumption. So graphically, the condition above can be understood as the downward sloping curve in $b_i$ defined by the LHS crossing the zero at the optimal choice of $b_i$.

2.2 Prudence and comparative statics on demand

First, we shall look at the property of individual demand. Or equivalently, we can write the problem as borrowing at a fixed real rate of interest. Also having the same vNM utility for both periods is not essential. To illustrate these points, let $r$ be the rate of interest (for safe loan) and write $v$ for $\delta_i u_i$. I shall omit reference to $i$. Then one would solve

$$\max_b u \left(m - b \right) + E \left[v \left((1 + r) b + Y \right)\right] \tag{2.3}$$

where $E$ is the expectation with respect to random variable $Y$.

This function is concave in $b$ thus the following first order condition is necessary and sufficient$^2$ for this problem is

$$-u' \left(m - b \right) + E \left[u' \left((1 + r) b + Y \right) \left(1 + r \right)\right] = 0. \tag{2.4}$$

$^1$On top of this, it is sensible to impose the constraints of non-negative consumption, but here I assume that the optimal consumption in the optimal problem above is positive and thus the positivity constraint is redundant.

$^2$There is a slight issue of differentiability: whether we can change the order of differentiation so that the derivative of the expectation is the same as the expectation of the derivative.
Notice that the left hand side is a decreasing function of $b$, since
\[
\frac{\partial}{\partial b} (-u'(m - b) + E[v'((1 + r)b + Y)(1 + r)]) \\
= u''(m - b) + E[v''((1 + r)b + Y)(1 + r)^2] \\
< 0.
\]

As a benchmark, let us study the case where the future income is not random. Let $Y = y_0$ for sure. Denote by $b_0$ the corresponding optimal saving level. Then re-writing the FOC (2.4) above for this special case we have:
\[
-u'(m - b_0) + E[v'((1 + r)b_0 + y_0)(1 + r)] = 0. \tag{2.5}
\]

Now let’s ask how the optimal saving level changes with respect to an increased uncertainty or risks in the future income. It will become clear that as far as the signs of comparative statics with respect to the riskiness of income are concerned, the rate of interest is not a factor, and we set $r = 0$ to simplify the notation.

Since we want to take the effect of decreased/increased income risks, we want to keep the mean of income fixed. For this purpose, suppose that $Y_0$ and $Y_1$ are random variables of the same mean and $Y_1$ is riskier than $Y_0$. That is, there is a random variable $Z$ whose conditional mean with respect to $Y_0$ is zero such that $Y_1 = Y_0 + Z$. In the special case we considered above, $Y_0$ is a constant $y_0$ for sure. Assume further $v'' > 0$ (i.e., $v'$ is convex). Let us write the optimal amount of saving in each problem by $b_0$ and $b_1$, respectively. So from the first order condition,
\[
-u'(m - b_0) + E[v'(b_0 + Y_0)] = 0 \\
-u'(m - b_1) + E[v'(b_1 + Y_1)] = 0
\]

I will show that $b_1 \geq b_0$, that is, you save more if future income risk increases. The motive for this additional saving is commonly referred to as the precautionary motive for saving.

Assume on the contrary that $b_1 < b_0$. Since $v'$ is decreasing, for any realization of $Y_0$, $v'(b_0 + Y_0) < v'(b_1 + Y_0)$, so we have $E[v'(b_0 + Y_0)] < E[v'(b_1 + Y_0)]$. Since $Y_1$ is riskier than $Y_0$, $b_1 + Y_1$ is riskier than $b_1 + Y_0$, and since $v'$ is convex (so it is like a risk loving agent), by Jensen’s inequality, we have $E[v'(b_1 + Y_0)] \leq E[v'(b_1 + Y_1)]$. Thus $E[v'(b_0 + Y_0)] < E[v'(b_1 + Y_1)]$. But from the first order conditions, this implies that $-u'(m - b_0) > -u'(m - b_1)$ and hence $b_0 < b_1$ since $u'$ is decreasing. A contradiction.
In the analysis above, the key condition is the prudence, that is, a positive third derivative of vNM utility function \( v_i \). The precautionary dissaving may occur if the agent is not prudent, i.e., the third derivative is negative somewhere. In many economic applications we do get precautionary saving rather than dissaving since the common functional form like CARA, CRRA, (and so the logarithm) exhibits prudence.

Finally I shall note that the effect of change in interest rate \( r \) is tricky. Say interest rate \( r \) increases. On the one hand, it makes the relative price of saving cheaper, which should induce more saving. On the other hand, since one unit of saving yields more income in future, you would not need as much saving to keep your consumption level smooth across time.

This intuition can been seen again by inspecting how the left hand side of (2.4) changes as \( r \) increases, keeping all the other parameters fixed. Differentiating the left hand side of (2.4) with respect to \( r \), we have \( E[v''((1+r)b+Y)(1+r)b]+E[v'((1+r)b+Y)] \), which cannot be signed clearly. By the same logic above, if this sum is positive, then saving must increase, and vice versa. The second term \( E[v'((1+r)b+Y)] \) is always positive, representing the direct effect of saving becoming cheaper. The first term is more to do with consumption smoothing. Notice that if \( b<0 \), i.e., you are borrowing to start with, the first term is positive, so saving increases (i.e., you reduce the amount you borrow).

### 2.3 General equilibrium

Now I shall turn to the analysis of markets. A general competitive equilibrium occurs when consumers are maximizing the utility function given prices and income, and the all markets clear (i.e. the quantity demanded equals the quantity supplied). There are many markets in this economy: the market for the bond, and the market for the consumption good in period 0 and that for in period 1. For instance, in the second period, the total consumption must be the same as the total supply of the good, which is \( \sum_i Y_i + B \) (notice that \( B \) is supplied by the bond issuer). Moreover, since \( Y_i \) is random, we have to make sure that the market clears for any realization of \( Y \). But fortunately, if the bond market clears, then any of those markets for the consumption good clears as well. To see this, suppose that the bond market clears, that is, \( \sum b_i = B \). Then the aggregate demand for the good in period 0 is \( \sum_i x_i^0 = \sum_i (e_i^0 - qb_i) = \sum_i e_i^0 - q \sum_i b_i^0 = \sum_i e_i^0 - qB \), which is equal to the supply of the good (notice that the bond issuer consumes \( qB \) - the issuer takes away \( qB \) units of the good in period 0 (perhaps use it for production) and pay back \( B \) units in period 1. In period 1, the supply of the good is \( \sum_i Y_i + B \) whereas the demand for the good is \( \sum_i X_i = \sum_i (b_i + Y_i) = \sum_i Y_i + B \); that is, the demand equals the supply at every state (i.e., probability one).

In summary, when we find price \( q \) which clears the bond market, then the supply meets the demand in all the other markets simultaneously, i.e., we have a general competitive equilibrium. Writing \( b_i(q) \) the solution to the above maximization problem, we have a general competitive equilibrium if and only if \( \sum_i b_i(q) = B \).

**Equilibrium comparative statics**: say that \( q^* \) is an equilibrium price and
suppose that the aggregate demand $\sum_i b_i(q)$ is decreasing in $q$ around $q^*$. Then, the equilibrium price increases (i.e., the interest rate decreases), if the following changes occur by a small amount:

- total supply $B$ decreases,
- future income for some households get more random (assuming the prudence condition) - because as we have seen it will increase the demand.

### 2.4 A Bond pricing formula

When $I = 1$ and $B = 0$, we obtain a simple pricing rule for the bond: equilibrium means the only (representative) consumer chooses $b_i = 0$ as a result of utility maximization. So, from the expression above, we find:

$$q^* = \frac{\delta \mathbb{E}[u'(Y)]}{u'(e^0)}$$

gives the equilibrium price of bond, thus the equilibrium rate of interest is $\frac{1}{q^*} - 1$.

From this formula, we see that when the consumer becomes more patient, i.e., the discount factor increases, the equilibrium $q^*$ increases. That is, if the agent gets more patient, the rate of interest goes down. This is intuitive, since patience will decrease the demand for borrowing, thus to induce more borrowing the interest rate must come down.

Now consider changes in the riskiness of $Y$: suppose that the second period endowment gets riskier, that is, the random variable $Y$ changes to $Y'$ and $Y'$ is a riskier random variable than $Y$ in the sense of the second order stochastic dominance (SOSD), then $\mathbb{E}[u'(Y)]$ may or may not increase, depending of the concavity of $u'$ (not $u$ itself). In the standard examples (such as CARA, log, etc), we have $u''' > 0$, which means that $u'$ is a convex function, thus $u'$ will behave like a risk lover. So $\mathbb{E}[u'(Y)]$ will **increase** if $Y$ gets **riskier**, and thus the bond price will increase, the rate of interest will decrease.

Since risk aversion by itself does not determine the sign of $u'''$, the results above should not be associated to risk aversion. It is common to interpret that $u'''$ measures the decision makers prudence. When $u''' > 0$, the agent is prudent and wants to save for a precautionary reason so when $Y$ gets riskier, he wants to save more, other things being equal. So the interest rate must go down to reduce saving in equilibrium.

### 2.5 Production

The set up so far does not explain how the risky income $Y_i$ and the supply of bond $B$. Presumably the fund is raised to finance production activity, and the income is a result of production activity and the distribution of profits. So here we shall look at a simple model where firms are explicitly considered.
2.5.1 Output shocks

Let there be $J$ firms. Each firm $i$ has production function $f_j$ and is subject to a random productivity factor $\eta_j$, which is a positive random variable: if $z_j$ units of the good is used as an input in period 0, then $\eta_j f_j(z_j)$ units of the good will be produced. By assumption, $f_j' > 0$ and $f_j'' < 0$. Since the scale for $f_j$ is arbitrary, we may as well normalize so that $\mathbb{E}[\eta_j] = 1$.

In order to purchase the good in the market, each firm needs to raise fund, and here they do it by selling a riskless bond promises to pay one unit of good in the second period. We assume that there will be no default, and so these bonds issued by different firms are perfect substitutes and thus all of them will have the same price, denoted by $q$ in units of the good. Let $B_j$ the amount firm $j$ sells in the bond market. Then firm $j$ acquires $qB_j$ in the bond market which it will use to purchase the good in the commodity market simultaneously. Thus firm $j$’s input is $qB_j$. In the second period, firm $j$ pays back $B_j$ units of good, and so $\eta_j f(z_j) - B_j$ is the realized profit.

Each firm is interested in maximizing the expected profits:

$$\max_{B_j} \mathbb{E}[\eta_j f_j(qB_j) - B_j]$$

but since $\eta_j$ is not affected by the choice of $B_j$, the problem above is nothing but (recall $\mathbb{E}[\eta_j] = 1$):

$$\max_{B_j} f_j(qB_j) - B_j$$

which is characterized by the first order condition.

$$q f_j'(qB_j) = 1.$$  

The profits are distributed to the consumers, who are the ultimate owners of the firms. Let $\theta_{ij}$ be consumer $i$’s share of firm $j$ profits. By definition $\sum_j \theta_{ij} = 1$ for every firm $j$, and assume that $\theta_{ij} \geq 0$. Thus given the firms activities, consumer $i$’s income will be

$$Y_i = \sum_{j=1}^J \theta_{ij} (\eta_j f_j(qB_j) - B_j).$$

From the consumer’s point of view, there is no difference from the previous set up. Each consumer takes prices given as well as the income $Y_i$ as defined above. Put it differently in the model of perfect competition, each consumer does not think he can influence the firm’s activities. So we can just use the previous setup for the consumers.

As before, a general equilibrium occurs when all the markets clear, i.e., consumers and firms are maximizing their respective objective functions given prices and income, the commodity markets as well as the bond market. But again, we need to check if the bond market clears, since all the other markets will clear automatically. To see this, suppose that the bond market clears, that is, $\sum_i b_i = \sum_j B_j$. Then the aggregate demand for the good in period 0 from the consumers is $\sum_i x_i^0 = \sum_i \theta_{ij} (\eta_j f_j(qB_j) - B_j)$.

\textsuperscript{3}Here implicitly the share holders have unlimited liability for the firms.
\[ \sum_i (e_i^0 - q b_i) = \sum_i e_i^0 - q \sum_i b_i^0 = \sum_i e_i^0 - q \sum_j B_j, \] which is equal to the supply of the good to the consumers (notice that each firm \( j \) uses \( q B_j \) thus the firms consume \( q \sum_j B_j \) units of the good. In period 1, the total production of the good is \( \sum_j \eta_j f_j (q B_j) \) which is supplied to the commodity market. On the other hand, the consumers demand \( \sum_i X_i \) in total. Now

\[
\sum_i X_i = \sum_i (b_i + Y_i) \quad \text{(budget constraint)}
\]

\[
= \sum_j B_j + \sum_i Y_i \quad \text{(bond market clearing)}
\]

\[
= \sum_j B_j + \sum_i \left( \sum_{j=1}^J \theta_{ij} (\eta_j f_j (q B_j) - B_j) \right) \quad \text{(definition of } Y_i \text{)}
\]

\[
= \sum_{j=1}^J \eta_j f_j (q B_j) \quad \text{(since } \sum_i \theta_{ij} = 1 \text{ for every } j),
\]

which confirms that the markets are clearing in the second period for sure.

### 2.5.2 Input shocks

Now assume that each household is endowed with another factor of production, labor, at the beginning of the second period. Assume that the labor yields neither utility nor disutility. So consumers will supply all the labor they have. But the labor is subject to some idiosyncratic productivity shock: let \( \zeta_i \) be a positive random variable and assume that consumer \( i \) is endowed with \( \zeta_i L \) units of labor. One may interpret for instance that every consumer has the same length of potential labor hours, but they differ in random productivity. Assume that \( \sum \zeta_i = 1 \) with probability one. That is, these risks are purely personal and averaged out in aggregation.

There is a labor market in the second period, in addition to the commodity markets as well as the bond market, and denote by \( w \) the wage rate in units of the good to be determined in the labor market. Since consumer \( i \) will supply the entire endowment of labor, his labor income is \( w \zeta_i L \). His income \( Y_i \) will be the sum of the labor income and the profits, but we shall make an assumption which implies that the profits are zero in equilibrium.

Each firm has a constant returns of scale technology, and produces the good in period from the input of the good in the first period and the labor input in the second period: write \( F_j (K, L) \) for the production function, which is concave and homogeneous of degree one. Here \( K \) refers to the amount of input in the first period, and \( L \) is the amount of labor in the second period. As before \( q B_j \) is the amount of input in period zero. Writing \( L_j \) for the labor input, the firm will produce \( F_j (q B_j, L_j) \), and so the firm’s profit is

\[
F_j (q B_j, L_j) - B_j - w L_j
\]

each firm will choose \( B_j \) and \( L_j \) to maximize profits. Because of the constant returns to scale, in equilibrium, every firm must be making zero profit (or else a firm can scale...
up the production to earn arbitrarily high profits), i.e., \( F_j(qB_j, L_j) - B_j - wL_j = 0 \) must hold. So as far as the description of equilibrium is concerned, we can omit the profits distribution to the consumers. The first order condition of optimization is:

\[
q \frac{\partial}{\partial K_j} F_j(qB_j, L_j) = 1 \\
\frac{\partial}{\partial L_j} F_j(qB_j, L_j) = w
\]

Thus consumer \( i \)'s second period income is \( Y_i = w\zeta_i \bar{L} \), and consumer \( i \)'s problem is defined as before.

A general equilibrium occurs when all the markets clear, as before. The labor market clearing says \( \sum_j L_j = \sum_i \zeta_i \bar{L} = \bar{L} \) where the last equation holds because \( \sum \zeta_i = 1 \) is assumed. The bond market clearing is \( \sum_i b_i = \sum_j B_j \). And if these two markets are clearing, the other commodity markets are all clearing, thanks to the zero profit condition \( F_j(qB_j, L_j) - B_j - wL_j = 0 \).

**Exercise 2.1** Show that the markets for the good clear if both the bond and the labor markets clear.
Chapter 3

General Equilibrium with Incomplete Financial Markets

The model discussed in the previous chapter is a special case of the so called general equilibrium model with incomplete financial markets (GEI). I shall describe a general framework of GEI.

3.1 Basic set up for pure exchange and Efficiency

Suppose that there is a finite number of underlying states. Denote by \( S \) the set of states and denote by \( s \in S \) its generic element. It is convenient to regard period 0 as \( s = 0 \). Suppose that there are \( L \) goods to be consumed at every state. So there are \( L = (S+1)L' \) goods. Write \( x^s = (x^{s_1}, ..., x^{s_L'}) \in \mathbb{R}^{L'}_+ \) for a vector of goods to be consumed when state \( s \) occurs. So \( x = (x^0, x^1, ..., x^S) \in (\mathbb{R}^{L'})^{S+1} \) describe a complete plan of consumption.

Each consumer \( i \) has a consumption set \( X_i \subset \mathbb{R}^{L'} \): by definition, it is the set of consumption plans which consumer \( i \) may choose if affordable. Consumer \( i \)'s preferences over consumption plans are represented by a utility function \( U_i \). Most of the time we shall assume \( X_i = \mathbb{R}^{L'}_+ \) or \( X_i = \mathbb{R}^{L'}_+ \) and \( U_i \) is a \( C^2 \) function on \( \mathbb{R}^{L'}_+ \). Consumer \( i \) is endowed with a bundle of goods \( e_i \) to be traded in the markets. Denote by \( \bar{e} \) the bundle of goods available in the economy, i.e., \( \bar{e} = \sum_{i=1}^I e_i \in \mathbb{R}^L \).

Example 3.1 The background risk model can be seen as an instance of this general framework. I shall illustrate it when \( B = 0 \). Let \( L' = 1 \). Denote by \( S \) the underlying state space for the random variables \( Y_i, i = 1, ..., I \). Then consumer \( i \) is endowed with \( Y_i(s) \) when \( s \) occurs. So the initial endowments vector is \( e_i = (e^0_i, Y_i) = (e^0_i, \cdots, Y_i(s), \cdots) \). Consumption is implicitly defined by the rule \( x^0_i = e_i - qb_i \) and \( x^s_i = Y_i(s) + b_i \).

Example 3.2 Linear constraints: \( X_i \) a linear translation of the linear subspace spanned by vectors \( a_1, ..., a_K \) in \( \mathbb{R}^L \) which are linearly dependent (plus positive consumption constraint), i.e., \( X_i = e_i + \left\{ \sum_{k=1}^K z_k a_k : z_k \in \mathbb{R} \right\} \) where \( e_i \in \mathbb{R}^L \). This often comes up in applications in finance: in interpretation, \( a_1, ..., a_K \) are...
basic assets and $X_i$ is the set of consumption bundles attainable by some portfolio of them. For this interpretation, it is convenient to assume that the elements of $a_k$ corresponding to state 0 are all zero. Alternatively, let $\pi_1, \ldots, \pi_{L-K}$ be a linear base for the orthogonal complement of $X_i$ and write $P$ for the matrix consisting of row vectors $\pi_1, \ldots, \pi_{L-K}$. Then $X_i = \{e_i + z \in \mathbb{R}^L : Pz = 0\}$.

**Definition 3.3** A profile of consumption bundles $x = (x_i)_{i=1}^I \in (\mathbb{R}^L)^I$ is feasible (relative to $(X_i)_{i=1}^I$) if $x_i \in X_i$ for every consumer $i$ and $\sum_{i=1}^I x_i = \bar{e} + \sum_{j=1}^J y_j$.

**Definition 3.4** A feasible consumption allocation $x$ is weakly Pareto efficient if there is no feasible consumption allocation $x'$ such that $U_i(x'_i) > U_i(x_i)$ for all $i$.

**Remark 3.5** A feasible consumption allocation $x$ is said to be Pareto efficient if there is no feasible consumption allocation $x'$ such that $U_i(x'_i) \geq U_i(x_i)$ for all $i$ and $> \text{ holds for at least one } i$. Pareto efficiency is logically stronger but in our analysis it is equivalent to the weak Pareto efficiency (since if $U_i(x'_i) > U_i(x_i)$, then a small among of a desirable good could be taken from this individual and then distributed to the others, making everybody strictly better off). Hence we shall use the word “efficient” to mean both.

**Remark 3.6** By definition, Pareto efficiency is meant to capture the idea of no wasteful use of scarce resources. Thus in particular, it does not necessarily indicate equity or fairness; efficiency does not mean that the goods are distributed in an ideal manner in any sense. For instance, if all the consumable goods are allocated to a single consumer, it is Pareto efficient because it will be impossible to increase the other consumers’ welfare without (albeit very slight) decreasing the single consumer’s welfare. A large number of people being very poor seems to be a bad state, but it may well be efficient in the sense that nobody is wasting resources.

The following well known characterization result will be useful later. The idea behind this result can easily be obtained by a graphic argument using the Edgeworth box for the special case of $K = L$ (thus $P = 0$): the condition says that indifference curves must be tangent to each other.

**Proposition 3.7** Let $X_i = \{e_i + z \in \mathbb{R}^L : Pz = 0\}$ for every $i$, where $P$ is an $(L-K) \times L$ matrix of rank $L-K$. Assume that utility functions are differentiable, strictly increasing and concave, and let $x \in (\mathbb{R}^L_+)^I$ be feasible (i.e., it is an interior point). Then $x$ is weakly Pareto efficient if and only if there are $\lambda_i > 0$, and $\mu_i \in \mathbb{R}^{L-K}$ for $i = 1, \ldots, I$, and $p \in \mathbb{R}^L_+$ such that $\lambda_i DU_i(x_i) = p + \mu_i P$, for every $i$ (vectors are row vectors). In particular, when $K = L$, a feasible $x$ is weakly Pareto efficient if and only if there are $\lambda_i > 0$, $i = 1, \ldots, I$, and $p \in \mathbb{R}^L_+$ such that $\lambda_i DU_i(x_i) = p$ for every $i$. (i.e., the gradient vectors must be all collinear).

The geometry for the result is simple for the case $K < L$: by construction, if $Pz = 0$, $(p + \mu_i P) \cdot z = 0$ holds if and only if $p \cdot z = 0$. Thus $\lambda_i DU_i(x_i) = p + \mu_i P$ for every $i$ means that although $DU_i(x_i)$ may not be collinear with each other, the projection of $DU_i(x_i)$ on $\{z : Pz = 0\}$ must be collinear with each other.
Proof. It can be readily shown that \( x = (x_1, ..., x_I) \) is weakly Pareto efficient if and only if there are \( \lambda_i > 0, \ i = 1, ..., I \), and \( z_i = x_i - e_i \) is a solution to the following maximization problem:

\[
\max_{z_1, ..., z_I} \sum_{i=1}^{I} \lambda_i U_i (e_i + z_i)
\]

subject to

\[
\sum_{i=1}^{I} z_i = 0
\]

\( Pz_i = 0 \) for every \( i \).

Writing \( p \in \mathbb{R}^L \) for the Lagrangian multipliers for \( \sum_{i=1}^{I} z_i = 0 \), and \( \mu_i \in \mathbb{R}^{L-K} \) for the Lagrangian multipliers for \( Pz_i = 0 \) for every \( i \), the Kuhn Tucker condition for this problem is:

\[
\lambda_i DU_i (x_i) = p + \mu_i P, \ i = 1, ..., I
\]

and the constraints. ■

Exercise 3.8 Prove the assertion in the proof about the relation between the weak Pareto efficiency and the optimization problem.

Exercise 3.9 Write the definition of feasible consumption and the Pareto Efficiency in the background risk model. Suppose in addition that \( \sum_i Y_i = \bar{y} \) for sure in the background risk model. Show that in any Pareto efficient allocation, consumption must be state independent (i.e., for every \( i \), \( x_i^s \) is independent of \( s \)).

3.2 Model of General Equilibrium in Incomplete Financial Markets

Here I introduce a model of general equilibrium with financial markets with two periods (often referred to as GEI model), standard in the literature.

The description of the model is as follows. There are 2 periods, 0, 1. There are \( S \) states, describing the underlying uncertainty, labeled by \( s = 1, ..., S \) in period 1. By convention, period 1 will be treated as state 0. In each period, \( L' \) perishable goods which are traded in competitive markets as follows:

Spot markets: Write \( p^s \in \mathbb{R}^{L'} \) for the prices of goods in state \( s \), and \( x_i^s \in X_i^s \subset \mathbb{R}^{L'} \) the consumption bundle to be consumed by consumer \( i \) in state \( s \). These markets are interpreted as spot markets, and prices \( p^s \) are referred to as spot prices. We will make assumptions so that prices are strictly positive in equilibrium. So assume \( p^s \in \mathbb{R}^{L'}_{++} \).

Financial markets: there are \( J \) assets (securities), competitively traded in period 0. Asset \( j \) yields \( r_j^s (p^s) \) in units of account (dollars).

\(^1\)I shall refer it as “dollar” but this should be distinguished from the usual meaning of “money".
• The following two types of assets are considered most of time.

  – real asset: it pays a bundle \( y_j^s \) of the goods in state \( s \), thus \( r_j^s(p^s) = p^s \cdot y_j^s \). This type of asset can be seen as a simplest possible way of describing a firm - the firm has set up production process in such a way that a bundle of good \( y_j^s \) will be realized. Or, this can be regarded as an instance of perfectly indexed security, whose real payout is fixed.

  – nominal (purely financial) asset: the yields are fixed in the unit of account. \( r_j^s(p^s) \equiv r_j^s \)

Write \( r_j(p) \) for the column vector of yields (returns) when spot prices are \( p \), i.e.,

\[
\begin{bmatrix}
\vdots \\
r_j^s(p^s) \\
\vdots \\
r_1^1(p^1) & \cdots & r_1^j(p^1) \\
\vdots & \ddots & \vdots \\
r_S^1(p^S) & \cdots & r_S^j(p^S)
\end{bmatrix}
\]

which is a \( S \times J \) matrix, called the return matrix (at prices \( p \)).

The price of asset \( j \) is denoted by \( q^j \), the vector of asset prices is written as \( q = (q^1, \ldots, q^J) \).

**Example 3.10** The background risk model continued. Recall that \( L' = 1 \) was assumed. There is one real asset, which pays one unit of the good in each state \( s \in S \). That is, \( r^s(p^s) = p^s \) at any \( s \).

**Example 3.11** (options) a call option with strike price \( c \) on some real asset with payoff bundles \( y = (\cdots, y^s, \cdots) \): \( r_j^s(p^s) = \max\{p^s \cdot y^s - c, 0\} \).

**Consumers** (traders): There are \( I \) consumers. Consumer \( i \) has preferences of consumption bundles (or a plan of consumption) which are described by a smooth, concave utility function \( U_i \). Consumer \( i \) is endowed with a vector \( e_i \in \mathbb{R}^L \) of consumption goods, initial holding of assets \( \overline{b}_i^j \). Unless otherwise noted, we assume that \( \overline{b}_i^1 = 0 \).

  • In most applications we assume it conforms with the expected utility hypothesis for temporal randomness, and write a vNM utility function \( u_i \). That is, the utility of a bundle \( x_i = (x_i^0, x_i^1, \ldots, x_i^S) \) is \( U_i(x) = \sum_{s=1}^S \mu_i^s u_i(x_i^0, x_i^s) \), where \( \mu_i^s > 0 \) is a subjective probability of state \( s \) held by consumer \( i \).

  • \( x_i \): vector of consumption plans. \( x_i = (\cdots, x_i^s, \cdots) \), \( x_i^s \in \mathbb{R}^{L'} \)

In fact, it is difficult to introduce “paper money” with positive net supply in a finite model like this, since such an object cannot have market value at all time: at the last period, nobody wants to hold it and so the value must be zero. Foreseeing this, nobody wants to hold it one period before, and this argument iterates. So here the presumption is that there is some agreed unit of account, which is used for all the transactions.
Matrix notation is useful to keep the notation simple.

- \( b_i \) : the amount of security \( j \) in the portfolio household \( i \) holds at the end of the first period (negative \( b_{ij} \) means short sales). Then the portfolio yields \( \sum_j r_j^s(p^s) b_{ij} \) in state \( s \).

**Remark 3.12** It is often assumed that the utility function is also time separable; that is, \( u_i(x_i^0, x_i^s) = u_i(x_i^0) + \delta u_i(x_i^s) \), where the constant \( \delta \in (0, 1) \) is called the discount factor. Notice in such a setup however \( \sum_s \mu_i^s u_i(x_i^0, x_i^s) = u_i(x_i^0) + \sum_s \delta \mu_i^s u_i(x_i^s) \), where the latter could be interpreted as if the second period utility is \( \delta u_i \) (i.e., it is a different utility function) with discount rate equals to one. This means that mathematically, the constant \( \delta \) and beliefs \( \mu_i \) are difficult to identify although they have different economic interpretation, unless we know the vNM function \( u_i \).

**Remark 3.13** Most of the fundamental results described later do not depend on the assumption of the expected utility. The reader will see that in most of the proofs only \( U_i \) is referred to. On the other hand, without expected utility assumption, there is some consistency problem in the definition of a competitive equilibrium to be explained later. In other words, the expected utility assumption is for simplicity and for the economic interpretation of the competitive equilibrium, but not for its technical merits.

Denote by \( \bar{b}_j = \sum_i \bar{b}_{ij} \), which is the total endowments of assets. Then when assets are all real, \( \sum_j y_j^s b_j \) is the total bundle of goods generated by the securities (assets) as a whole. Unless noted otherwise we shall assume that the net supply of assets is zero, \( \sum_i \bar{b}_{ij} = 0 \).

**Remark 3.14** It is common to assume, and we do so too in the following, that \( \sum i \bar{b}_{ij} = 0 \) when the assets are nominal (since it is hard to justify a positive value for a purely nominal object with a positive net supply - how could it have a positive value!)
So consumer’s problem is, taking all the prices (and endowments) as given, to solve:

\[
\begin{align*}
\max_{x_i, b_i} U_i \left( x^0_i, x^1_i, \ldots, x^S_i \right) \\
\text{subject to} \\
p^0 \cdot x^0_i + \sum_j q_j b_{ij} = p^0 \cdot e^0_i + \sum_j q_j \bar{b}_{ij} \\
p^s \cdot (x^s_i - e^s_i) = \sum_j r^s_j (p^s) b_{ij} \text{ for each } s = 1, \ldots, S.
\end{align*}
\]  

(3.1) (3.2)

Definition of a competitive equilibrium and its basic properties.

Now we are ready to proceed to give a definition of competitive equilibria.

**Definition 3.15 (Competitive equilibrium)** A competitive equilibrium for the GEI model (Rational expectation equilibrium, Radner equilibrium, Arrow-Radner equilibrium): \((p^*, q^*, x^*, b^*)\) is an AR equilibrium if for each \(i\), \((x^*_i, b^*_i)\) solves the maximization problem as above, and all the markets clear: i.e., \(\sum_i x^0_i = \sum_i e^0_i\), \(\sum_i x^s_i = \sum_i e^s_i + \sum_i (\sum_j y^s_j \bar{b}_{ij})\), for \(s = 1, \ldots, S\), and \(\sum_i b_{ij} = \bar{b}_j\) for \(j = 1, \ldots, J\). (and \(\bar{b}_j = 0\) for \(j = 1, \ldots, J\) is assumed for the case of nominal assets).

Notice the role of “rational” or self-fulfilling expectation. When the first period trade takes place, the second period trade has yet to occur. Thus at that point \(x^s\) is just a description of a plan of trading. The equilibrium concept requires that the expected trading plans in fact clear the markets. This point can be seen more clearly if we consider the nominal price of good. It is important that the expected nominal prices are the same across the traders.

**Remark 3.16** it is often assumed that there is no consumption in period 0. In such a case, the budget constraint for period 0 is simply \(\sum_j q_j b_{ij} = \sum_j q_j \bar{b}_{ij}\).

There are two immediate but important properties of equilibria. The first is called the **Walras law**:

**Lemma 3.17** Walras law: assume that the commodity prices are positive. When asset markets clear as well as \(L' - 1\) good markets clear in every state, then the rest of the good markets clear automatically.

**Proof.** In period 0, from the budget constraint, \(\sum_i \left( p^0 \cdot x_i^0 + \sum_j q_j b_{ij} \right) = \sum_i \left( p^0 \cdot e_i^0 + \sum_j q_j \bar{b}_{ij} \right)\), and so the asset market clearing (i.e., \(\sum_i b_{ij} = \sum_i \bar{b}_{ij}\)) implies that \(p^0 \sum_i (x_i^0 - e_i^0) = 0\). By assumption, \(\sum_i (x_i^0 - e_i^0)\) is a vector with at most one non-zero element, so this equation means that \(\sum_i (x_i^0 - e_i^0)\) must be the zero vector. Similarly, in state \(s\), from the budget constraint and the asset market clearing we get \(p^s \cdot \sum_i (x_i^s - e_i^s) = 0\), and so the same logic as above applies. ■

The second is the **homogeneity**, but this property depends on the type of assets:
Lemma 3.18 [Homogeneity for the real asset model] Assume that all the assets are real. Then the equilibrium is determined up to relative prices in each state. Thus there is no loss of generality if we normalize the price of one of the good equal to one at every state.

Proof. The budget constraint in each state is homogeneous in the prices, thus the consumers’ behavior is invariant if prices are multiplied by a positive scaler in state $s$, for any $s$. Hence the demand is determined by the relative prices in each state, and consequently the equilibrium is determined essentially by a system of relative prices in each state.

Example 3.19 In the background risk model, we set $p^s = 1$ for every $s = 0, 1, \ldots, S$. The homogeneity result shows that this is done without loss of generality.

Remark 3.20 Indeterminacy for the nominal asset model: on the other hand, for nominal assets, the budget is no longer homogeneous, and so the relative prices across the states might matter. Indeed, it does. We will see this later in detail, but the intuition can be provided here. Since the real value of a nominal asset depends on the commodity price, the randomness of the prices (i.e., uncertain inflation rates) changes the riskiness of the asset, hence induces different demand/supply.

3.3 No Arbitrage condition and State Prices

It is clear that under the standard monotonicity assumption on the utility function, equilibrium prices of goods should be positive. What do we know about asset prices? Since returns are uncertain and correlated with each other, we can expect that the equilibrium prices of assets will delicately depend on how their returns are evaluated by the consumers. But first, notice that the price of a redundant asset must be determined by the law of one price: an asset is said to be redundant (given $p$) if its payoffs can be expressed by a linear combination of other assets’ payoffs.

Lemma 3.21 Let $(p^*, q^*, x^*, b^*)$ be an equilibrium and suppose that asset $\bar{j}$’s payoffs can be expressed by a linear combination of other assets’ payoffs: $r_{\bar{j}}(p^*) = \sum_{k \neq \bar{j}} r_k(p^*) z_k$. Then the price of asset $\bar{j}$ must be given by $q_{\bar{j}}^* = \sum_{k \neq \bar{j}} z_k q_k^*$.

Proof. If $r_{\bar{j}}(p^*) = \sum_{k \neq \bar{j}} r_k(p^*) z_k$, then one unit of asset $\bar{j}$ and a portfolio of holding $z_k$ units of asset $k$ for each $k \neq \bar{j}$ have exactly the same return in every state (i.e., they are the same thing). Thus these two must have the same market value, and the market value of the latter is $\sum_{k \neq \bar{j}} z_k q_k^*$.

Example 3.22 [option pricing formula] Suppose there are two states, and the price of an asset which pays $(r^1, r^2)$, $r^1 > r^2$ is $q$. Also suppose that there is a riskless bond, which pays $(1, 1)$ and whose price is $\frac{1}{1+i}$ (i.e., the rate of interest is $i$). Then the price of a call option with strike price $c$ on the asset can be computed by the law of one price: say that $r^1 > c > r^2$, and so the return of the option is $(r^1 - c, 0)$. Solving $(r^1 - c, 0) = t_1 (r^1, r^2) + t_2 (1, 1)$, we find $t_1 = \frac{r^1 - c}{r^1 - r^2}$ and $t_2 = -\frac{r^2 - c}{r^1 - r^2}$. By Lemma 3.21, the price of the option must be $t_1 q + t_2 \frac{1}{1+i} = \frac{r^1 - c}{r^1 - r^2} (q - \frac{r^2}{1+i})$. 19
But we can infer more than this, because the monotonicity of preferences does have one obvious implication: asset prices must not admit a free lunch, i.e., there should not be a trading opportunity which earns positive income in every state without cost. Formally, asset prices $q$ must satisfy the following “no free lunch” condition given prices $p$:

$$
\text{for any } b_i, \text{ if } R(p) b_i > 0, \text{ then } q \cdot b_i > 0
$$

(3.3)

It turns out that the no free lunch condition (3.3) can be expressed in its dual form:

**Proposition 3.23** No free lunch condition (3.3) is satisfied if and only if there is a positive vector $\lambda \in \mathbb{R}^S_{++}$ such that $q = (R(p))^T \lambda$. (3.4)

That is, for each asset $j$, $q_j = \sum_{s=1}^{S} \lambda^s r_j^s (p)$ holds.

**Proof.** Suppose that there is a positive vector $\lambda \in \mathbb{R}^S_{++}$ such that $q = (R(p))^T \lambda$. Then $R(p) b_i > 0$ implies that $q \cdot b_i = \lambda^T (R(p) b_i) > 0$, which shows that (3.3) is satisfied.

Conversely, suppose that no free lunch condition (3.3) is satisfied. Note that (3.3) holds if some of the assets are eliminated, and once a vector $\lambda$ is found then the price of a redundant asset is given by $\sum_{s=1}^{S} \lambda^s r_j^s (p)$ by Lemma 3.21. So we may as well assume that all the columns of $R(p)$ are linearly independent.

Consider now a set $A := \left\{ \begin{bmatrix} -q \cdot b \\ R(p) b \end{bmatrix} : b \in \mathbb{R}^J \right\}$. It is readily verified that $A$ is a closed convex set in $\mathbb{R}^S$. Also, $A \cap \mathbb{R}^S_{++} = \{0\}$ must hold, or else (3.3) is violated (notice that $R(p) b = 0$ implies $b = 0$ since the columns of $R(p)$ are linearly independent). By the separation theorem (see Corollary 10.3), there is $\alpha > 0$ and $\hat{\lambda} \in \mathbb{R}^S_{++}$ such that $\alpha (-q \cdot b) + \hat{\lambda} \cdot (R(p) b) = \left(-\alpha q + (R(p))^T \hat{\lambda}\right) \cdot b \leq 0$ for any $b \in \mathbb{R}^J$. This is possible only if $-\alpha q + (R(p))^T \hat{\lambda} = 0$. So we have found a desired vector $\lambda := \frac{1}{\alpha} \hat{\lambda}$. ■

We shall refer to the relation $q = (R(p))^T \lambda$ as the no arbitrage condition, or no arbitrage asset pricing equation. The no arbitrage condition has an interesting economic interpretation. Imagine that $\lambda^s$ above is the period 0 price of one dollar in state $s$. That is, $\lambda^s$ is the discounted present value of a contract which promises to pay one dollar when state $s$ occurs. Then $q = (R(p))^T \lambda$, thus asset price equation $q_j = \lambda \cdot r_j (p)$ says that the price of asset $j$ is the present value of the future returns of asset $j$. For this reason, $\lambda$ appearing in the no arbitrage condition is referred to as the state prices (associated with commodity prices $p$ and asset prices $q$). To put it differently, we have shown that in equilibrium, asset prices must be determined so that state prices exist.

---

2For a vector $v$, $v > 0$ means that $v$ is non-negative and at least one element of $v$ is positive.
It is often convenient to normalize state prices: let \( \lambda \) be state prices and set \( Q^s = \lambda^s / \left( \sum_{s=1}^{S} \lambda^s \right) \). Then \( Q = (\cdots, Q^s, \cdots) \) is a probability measure on states, and the pricing equation (3.4) now reads as follows:

\[
q_j = \tilde{\lambda} \sum_{s=1}^{S} r_j^s (p^s) = \tilde{\lambda} E_{Q_i} \left[ r_j^s (p) \right] \tag{3.5}
\]

where \( \tilde{\lambda} = \sum_{s=1}^{S} \lambda^s \). The second equation is just re-writing of the first expression, emphasizing that \( Q_i \) is a probability measure. Notice that by construction, \( \tilde{\lambda} \) is the price of nominal riskless bond. So the equation (3.5) says that the price of asset \( j \) relative to the price of risk free rate of interest is the expected value of asset returns, where the expectation is taken with respect to measure \( Q \). For this reason, the measure \( Q \) is called the pricing kernel.

**Example 3.24 (option pricing formula, continued)** Recall the setup in Example 3.22. It can be solved by finding the pricing measure: from the riskless bond, \( \tilde{\lambda}_i = \frac{1}{1+i} \), and so the pricing measure must solve \( q (1+i) = Q^1 r_1 + (1-Q^1) r^2 \) hence \( Q^1 = \frac{q(1+i)-r^2}{r_1-r^2} (r_1-c) \). The price of the option should then be \( \frac{1}{1+i} \left( Q^1 \left( r_1-c \right) \right) = \frac{q(1+i)-r^2}{r_1-r^2} (q - \frac{r^2}{1+i}) \), which is exactly the same as the price found in Example 3.22.

### 3.4 State Prices and Individual Demand

The state prices have an interesting welfare implication, which we shall explore later. Here we shall study its role in the budget and the individual utility maximization problem.

**Lemma 3.25** Assume zero initial asset holdings. Suppose that given \( p, q \) satisfies the no arbitrage condition: \( q = (R(p))^T \lambda \) for some vector \( \lambda \in \mathbb{R}^S_{++} \). Set \( \tilde{p}^0 = p^0 \) and \( \tilde{p}^s = \lambda^s p^s \) for \( s = 1, \ldots, S \). Then, \( x_i \) is budget feasible consumption vector if and only if \( x_i \) satisfies the following auxiliary budget constraint:

\[
p^0 \cdot (x_i^0 - e_i^0) + \sum_{s=1}^{S} \tilde{p}^s \cdot (x_i^s - e_i^s) = 0 \tag{3.6}
\]

\[
(p^s \cdot (x_i^s - e_i^s))_{s=1}^{S} = R(p) b_i \text{ for some } b_i
\]

**Proof.** Suppose that \( x_i \) is budget feasible, and let \( b_i \) be the associated choice of asset portfolio. Then the second condition for the auxiliary budget constraint is trivially satisfied. For the first, note that the no arbitrage condition says that \( q^T = \sum_{s=1}^{S} \lambda^s r^s (p^s) \). Then multiplying by \( \lambda^s \) the state \( s \) budget and substituting them, we have \( p^0 \cdot (x_i^0 - e_i^0) + \sum_{s=1}^{S} \tilde{p}^s \cdot (x_i^s - e_i^s) = p^0 \cdot (x_i^0 - e_i^0) + \sum_{s=1}^{S} \lambda^s r^s (p^s) b_i = p^0 \cdot (x_i^0 - e_i^0) + q \cdot b_i = 0 \).
Conversely, suppose that \( x_i \) satisfies the auxiliary budget, and let \( b_i \) be the associated vector found in the second condition. By construction the state \( s \) budget is satisfied for \( s = 1, \ldots, S \). For \( s = 0 \), as we have seen above, \( p^0 \cdot (x_i^0 - e_i^0) + q_i \cdot b_i = p^0 \cdot (x_i^0 - e_i^0) + \sum_{s=1}^{S} \hat{p}^s \cdot (x_i^s - e_i^s) \).

Notice that the auxiliary budget constraint is determined by the (state price adjusted) commodity prices and the (column) span of the return matrix by construction. Lemma 3.25 therefore implies that a consumer’s consumption behavior is invariant if asset payoffs are changed in such a way that the span remains the same. So an equilibrium consumption can be characterized by the asset span, not a particular form of asset payoffs. We shall show this formally in the following.

**Lemma 3.26** Assume zero initial asset holdings. Given prices \( p \) of goods, suppose that two asset return matrices \( R \) and \( \hat{R} \) have the same linear span at \( p \). Suppose also that the no arbitrage condition holds for asset prices \( q \) for \( R(p) \): \( q = (R(p))^T \lambda \) for \( \lambda \in \mathbb{R}^S_+ \). Then \((x,b)\) satisfies the budget constraint for asset structure \( R(p) \) if and only if \((\hat{x},\hat{b})\) satisfies the budget constraint for asset structure \( \hat{R}(p) \) with asset prices \( \hat{q} = \left( \hat{R}(p) \right)^T \lambda \).

**Proof.** This follows immediately from Lemma 3.25.

**Proposition 3.27** Assume zero initial asset holdings. Let \((p^*, q^*, x^*, b^*)\) be an equilibrium. Consider another asset structure \( \hat{R} \) such that the span of \( \hat{R}(p^*) \) coincides with that of \( R(p^*) \). Then in the model where assets are given by \( \hat{R} \), there are \( \hat{q} \) and \( \hat{b} \) such that \( (p^*, \hat{q}, x^*, \hat{b}) \) constitute an equilibrium.

**Proof.** Since no free lunch is possible in equilibrium, by Proposition 3.23, \( q^* = (R(p^*))^T \lambda \) for some \( \lambda \in \mathbb{R}^S_+ \). Select linearly independent columns of \( \hat{R}(p^*) \), and for each \( i \), let \( \hat{b}_i \) be a unique linear combination of those columns such that \( (p^* \cdot (x^* - e_i^*)) = \hat{R}(p^*) \hat{b}_i \); this is possible since the span of \( R(p^*) \) coincides with that of \( \hat{R}(p^*) \) and by the definition of an equilibrium, \( (p^* \cdot (x^* - e_i^*)) \) must be in the span of \( \hat{R}(p^*) \). Setting \( \hat{q} = \left( \hat{R}(p^*) \right)^T \lambda \), Lemma 3.26 assures that the consumers’ choices remains the same for all the consumption goods. Thus, \( x_i^* \) is utility maximizing given \( p^*, \hat{q} \) and \( \hat{R}(p^*) \). The market clearing conditions for goods are satisfied by construction. For the asset markets, notice that by construction \( \sum_i \hat{R}(p^*) \hat{b}_i = \sum_i R(p^*) b_i^* \), so from \( \sum_i b_i^* = 0 \) we have \( \hat{R}(p^*) \sum_i \hat{b}_i = 0 \). Since \( \hat{R}(p^*) \sum_i b_i^* \) is a linear combination of linearly independent columns vectors, it implies that \( \sum_i \hat{b}_i = 0 \), so the asset markets clear as well.

An immediate implication of Proposition 3.27 is that if we add a redundant asset in the model, there will be an equilibrium with the same consumption allocation. In fact, there will be an equilibrium where the redundant asset is not traded (i.e., each consumer chooses to hold 0 unit of the asset). Indeed, given prices \( p \), suppose that asset \( j \)’s payoffs can be expressed by a linear combination of other assets’ payoffs: \( r_j(p) = \sum_{k \neq j} r_k(p) z_k \). Then if \((x_i, b_i)\) satisfies the budget for consumer \( i \), then
there is \( \hat{b}_i \) with \( \hat{b}_{ij} = 0 \) such that \( (x_i, \hat{b}_i) \) satisfies budget. If \( (x_i, b_i) \) satisfies the budget for consumer \( i \) and \( r_j^s(p) = \sum_{k \neq j} r_k(p) z_k \), then set \( \hat{b}_{ik} = b_{ik} + z_k b_{ij} \) for \( k \neq j \) and \( \hat{b}_{ij} = 0 \). Then by construction, in every state \( s \), we have

\[
\sum_j r_j^s(p) \hat{b}_{ij} = \sum_j r_j^s(p) b_{ij} - r_j^s(p) b_{ij} + \sum_{k \neq j} r_k^s(p) z_k b_{ij} = \sum_j r_j^s(p) b_{ij}.
\]

Thus \( (x_i, \hat{b}_i) \) meets the budget constraint in (3.1).
Chapter 4

Asset Pricing and Efficiency

The eventual purpose of this chapter is to establish a link between the asset pricing rule and the efficiency property of the equilibrium allocation. Here I assume that there is no restriction on consumption except for non-negativity, i.e., $X_i = \mathbb{R}^L_+$ for every $i$ for simplicity, but the argument readily extends for the case of linear constraints.

Assigning a Lagrangian multiplier $\lambda^s_i$, $s = 0, 1, ..., S$ to each of $S + 1$ equations in the consumer’s problem (3.1) above. Then the first order necessary and sufficient condition for an interior solution to this problem consists of: for each $i$,

$$\frac{\partial U_i}{\partial x^s_i} = \lambda^s_i p^s, \ s = 0, 1, ..., S. \quad (4.1)$$

or equivalently,

$$\lambda^0_i (q^1, \ldots, q^I) = (\lambda^1_i, \ldots, \lambda^S_i) R(p) \quad (4.2)$$

and the budget constraint (3.2).

Notice that the relation (4.3) says that the Lagrange multipliers normalized by $\lambda^0_i$ are state prices (recall 3.4). So we can also construct the corresponding (personalized) pricing kernel. Let $\bar{\lambda}_i := \sum_s \lambda^s_i$, and set $Q^s_i := \lambda^s_i / \bar{\lambda}_i$. Then $\sum_s Q^s_i = 1$ and so $\{Q^s_i : s \in S\}$ can be regarded as a probability measure on states $S$. Rewriting (4.3), the price $q_j$ of asset $j$ is

$$q^j = \frac{\bar{\lambda}_i}{\lambda^0_i} \sum_{s=1}^S \lambda^s_i r^s_j (p) \quad (4.3)$$

$$= \frac{\bar{\lambda}_i}{\lambda^0_i} \mathbb{E}_{Q^s_i} \left[ r^s_j (p) \right] \quad (4.5)$$

where $r^s_j (p)$ is the random variable of asset $j$’s returns. Then (4.4) (or equivalently (4.5)) says that when a consumer is optimizing, it must look as if the price of asset $j$ is the expected return of the asset with respect to this $Q_i$ measure, discounted by the factor $\frac{\bar{\lambda}_i}{\lambda^0_i}$.
The fact that the Lagrange multipliers constitute state prices are not very surprising. After all, the existence of state prices is equivalent to the no free lunch condition (Proposition 3.23), and the no free lunch condition is an obvious consequence of utility maximization and monotonicity of preferences. As we have seen before, the state price of state $s$ is the present value of one dollar in state $s$, which should be proportional to the marginal utility of income in state $s$. A more interesting question is whether or not these personalized state prices coincide or not, which we shall investigate in the next section. Here we conclude a straightforward observation about the coincidence of personal state prices.

Remark 4.1 The measure $Q_i$ is uniquely associated with the multipliers $(\lambda_s^i, s = 0, 1, ..., S)$, and it is invariant to scaling. That is, if for two consumer $i$ and $i'$, the vectors $(\lambda_0^i, \lambda_1^i, ..., \lambda_S^i)$ and $(\lambda_0^{i'}, \lambda_1^{i'}, ..., \lambda_S^{i'})$ are collinear, then the resulting measure coincide, i.e., $Q_i = Q_{i'}$, and vice versa.

Example 4.2 Suppose there is one good in each state, so the commodity space is $\mathbb{R}^{S+1}$. The price of an indexed bond, which pays off one unit of good for sure, is $\frac{\lambda_0^i}{X_i^0} \sum_s Q_i^s = \frac{\lambda_0}{X_i^0}$. In equilibrium, this must be independent of $i$.

It is interesting to ask the converse as well: stating with state prices implied by the no arbitrage condition, how are they related to the personal state prices, i.e., the Lagrange multipliers for utility maximization? To see this, fix prices for goods and assets, as well as state prices $\bar{\lambda}$ conforming with no free lunch, and compare the first order condition of the utility maximization with the auxiliary budget constraint (3.6), which is as follows: assigning multiplier $\lambda_0^i$ to the first constraint, and multiplier $\lambda_0^i \xi^s$ to $(p^s \cdot (x_i^s - e_i^s))_{s=1}^S = R(p) b_i$ for $s = 1, ..., S$,

$$\frac{\partial U_i}{\partial x_0^i} = \lambda_0^0 p^0,$$
$$\frac{\partial U_i}{\partial x_s^i} = \lambda_0^0 (\bar{\lambda}^s + \xi^s) p^s, \ s = 1, ..., S$$

and the budget constraint (3.6). The last equation shows that vector $(\bar{\lambda}^s + \xi^s)_{s=1}^S$ satisfies the no arbitrage condition for asset prices. Thus writing $\lambda_s^i := \lambda_0^0 (\bar{\lambda}^s + \xi^s)$, this set of equation is equivalent to the first order condition above. This is intuitive: personal state prices must respect the no arbitrage condition, and so it can differ from “market” state prices by $\xi$ which is perpendicular to $R(p)$ so that it does not affect the valuation of assets.

Now we are ready to establish the main result of this part:

Proposition 4.3 An (interior) equilibrium is Pareto efficient if and only if the associated pricing measures $Q_i, \ i = 1, ..., I$ are common across the consumers.
Proof. This is in fact a simple consequence of Lemma 3.7 and Remark 4.1. From Lemma 3.7, since $X_i = \mathbb{R}^L_i$ for every $i$, an equilibrium allocation is Pareto efficient if and only if $DU_i = \left(\frac{\partial U_i}{\partial x^0_i}, \frac{\partial U_i}{\partial x^1_i}, \ldots, \frac{\partial U_i}{\partial x^S_i}\right)$ are all collinear, which is to say $DU_i = \sigma_i \vec{p}$ for every $i$, where $\sigma_i > 0$ is a scaler and $\vec{p} = (\vec{p}^0, \vec{p}^1, \ldots, \vec{p}^S) \in \mathbb{R}^L_+$, and without loss of generality we may as well assume $\sigma_1 = 1$.

Suppose an equilibrium is efficient, and choose $\vec{p}$ and $\sigma_i$ as above. From the first set of first order conditions (6.4) of utility maximization, we have $\frac{\partial U_i}{\partial x^s_i} = \lambda^s_i \vec{p}$ for every $i$ and $s$. This means that spot prices are in fact must be collinear with $\vec{p}$ at every spot; indeed, looking at consumer 1, at every state $s$, $\frac{\partial U_1}{\partial x^s_1} = \vec{p}$ by construction, and so $\lambda^s_1 \vec{p} = \vec{p}$ must hold. Therefore by (6.4), for each $i$, $\frac{\partial U_i}{\partial x^s_i} = \lambda^s_i \vec{p}$ and hence we have $\sigma_i \vec{p} = \left(\frac{\lambda^s_i}{\lambda^s_1}\right) \vec{p}$ for every $s$, which occur if and only if $\lambda^s_i = \sigma_i \lambda^s_1$ for every $s$, since $\vec{p}$ is a non zero vector. Thus Pareto efficiency implies that the vector of multipliers $(\lambda^0_1, \lambda^1_1, \ldots, \lambda^S_1)$ are collinear with each other for all $i$, hence the measures $Q_i$ must coincide with each other by Remark 4.1.

Conversely, if the measures $Q_i$ coincide with each other, then by Remark 4.1, we may assume that $(\lambda^0_1, \lambda^1_1, \ldots, \lambda^S_1) = \sigma_i (\lambda^0_1, \lambda^1_1, \ldots, \lambda^S_1)$ for some $\sigma_i > 0$ for every $i = 2, \ldots, I$. Set $\vec{p} = \lambda^s_1 \vec{p}$ for every $s = 0, 1, \ldots, S$. From the first part of the first order condition (6.4), we then have $\left(\frac{\partial U_i}{\partial x^s_i}, \frac{\partial U_i}{\partial x^1_i}, \ldots, \frac{\partial U_i}{\partial x^S_i}\right) = \sigma_i (\vec{p}^0, \vec{p}^1, \ldots, \vec{p}^S)$ for every $i$, which implies efficiency.

Remark 4.4 For the case of the linear constraint model (recall Proposition (3.7)), an analogous result can be readily obtained.

Exercise 4.5 For the case of the linear constraint model, show that an equilibrium is efficient if and only if a suitable “projection” of $Q_i$ coincides with each other.
Chapter 5

Characterization of market Completeness

In the GEI setting the goods are not traded simultaneously, i.e., not all the good markets are open at the same time. But when there are a sufficient number of assets, one can effectively trade all the goods, and hence there is virtually a complete set of markets. The purpose of this chapter is to characterize the market completeness, that is, we ask in what sense the number of assets needs to be sufficient.

From now on, I shall assume that the endowments of assets are zero. This is done mostly for notational simplicity. There is no loss of generality for the case of real assets: for the case of real assets, the endowments of real assets can be translated into the second period endowments of goods. For nominal assets, we have already assumed the net total endowment is zero. Then the assumption of zero asset endowments means that the consumers have no financial credits nor obligations at the beginning of the first trading day.

5.1 Arrow Debreu Equilibrium

We have learned that the efficiency of equilibrium is not only important because of its normative appeal, but also because of its asset pricing implication. Then when do the markets produce an efficient allocation of consumption goods? To study this question, we first consider a model where all the contingent goods can be traded before any consumption takes place. This is obviously a fictitious construction, but it nonetheless provides a useful benchmark.

Let $p^s \in \mathbb{R}^{L'}_+$, $p^s = (p^{s1}, ..., p^{sL'})$, be a vector of prices for the contingent goods to be delivered/consumed in state $s$. Write $p = (p^0, p^1, ..., p^S) \in \mathbb{R}^L_+$ for the vector of those price vectors. By assumption, although the contingent goods are basically promises to be fulfilled in the future, their prices $p$ are determined in the contingent commodity markets which open before period 0. Consumers participate in the markets, taking these prices as given.

Suppose moreover that each consumer’s consumption is constrained in a translation of a linear subspace of $\mathbb{R}^L$ which contains the initial endowments. That is, there is a $c_i \times L$ matrix $C_i$ of rank $c_i$ and a consumer $i$ must choose a consumption
plan in the “consumption” set \( X_i := \{ e_i + z : C_i z = 0 \} \cap \mathbb{R}^L_i \). (By convention, \( e_i = 0 \) refers to the zero matrix, in which case there is no additional constraint. Consumer \( i \)'s budget is given by a single linear equation:

\[
x_i \in X_i \text{ and } p \cdot x_i = p \cdot e_i,
\]

which can also be written as:

\[
x_i \in X_i \text{ and } p^0 \cdot (x_i^0 - e_i^0) + \sum_{s \in S} p^s \cdot (x_i^s - e_i^s) = 0.
\]

A pair of a consumption allocation and a vector of prices \( (x^*, p^*) \in (\mathbb{R}^L_+)^I \times (\mathbb{R}^L_+^+) \) constitutes an Arrow Debreu competitive equilibrium if

1. **Utility maximization.** each consumer maximizes utility: \( x_i^* \in X_i \text{ and } p^* \cdot x_i^* = p^* \cdot e_i \). Also \( u_i (x) > u_i (x_i^*) \) and \( x \in X_i \) implies \( p^* \cdot x > p^* \cdot e_i \).

2. **Market clearing:** \( \sum_i x_i^* = \bar{e} \)

**Proposition 5.1** *The first fundamental theorem of Welfare economics.* Any Arrow Debreu competitive equilibrium allocation of goods is weakly Pareto efficient.

**Proof.** Let \( (x^*, p^*) \) be a competitive equilibrium. Suppose there is another feasible consumption allocation \( x \) that dominates \( x^* \), that is: for every consumer \( i \),

\[
u_i (x_i) > u_i (x_i^*)
\]

holds. We shall show that this leads to a contradiction.

Since \( x_i^* \) is utility maximizing, the inequality above must mean that \( x_i \) cannot be affordable, that is

\[
p^* \cdot x_i > p^* \cdot e_i
\]

holds for every \( i \). Summing these up, we have

\[
p^* \cdot \sum_i x_i > p^* \cdot \sum_i e_i = p^* \cdot \bar{e},
\]

which is inconsistent with the feasibility condition \( \sum_i x_i = \bar{e} \).

i.e., Edgeworth Box: \( S = 2, I = 2, L = 2 \) and there is no period for 0 consumption.

---

![Diagram](image-url)
\begin{align*}
u_1(x_1') > u_1(\bar{x}_1), \quad u_2(x_2') > u_2(\bar{x}_2). \quad \bar{x} \text{ is Pareto efficient.}
\end{align*}

Assume interior point \(x\). Then, an allocation is efficient if and only if there is \(p \in \mathbb{R}_+^L\) and weights \(\lambda_1, \ldots, \lambda_I > 0\) such that \(D\bar{u}_i(x_i) = \lambda_i \cdot p\).

Remark 5.2 For interior points, the result can be established via Lemma 3.7.

Remark 5.3 Since Pareto efficiency does not necessarily implies equity or fairness, the first fundamental theorem must be understood as such. For instance, a consumer who is happened to be endowed with a small amount of resources, he tends to receive less in equilibrium, and this is consistent with Pareto efficiency.

In an Arrow Debreu equilibrium, the price of an asset must be the same as the Arrow - Debreu market value of its payoffs by the law of one price (also known as no-arbitrage principle): the price of an asset whose return is a vector \(R \in \mathbb{R}^L\), its price must be \(p^* \cdot R\).

Example 5.4 Suppose there is one good in each state, so the commodity space is \(\mathbb{R}^{S+1}\). Denote AD equilibrium prices by \(p^* = (\cdots, p_s^*, \cdots)\). Then for instance the price of an indexed bond, which pays off one unit of good for sure, is \(\sum_{s \in S} p_s^*\) in period 0.

5.2 Complete Markets in GEI.

As we have seen before, the efficiency of a competitive in GEI can be characterized by the uniqueness of the pricing measure. The following simple observation, which is just a restatement of Proposition 4.3, is useful.

Lemma 5.5 If \(q^*\) are equilibrium asset prices, then there must exist a measure \(Q\) on \(S\) and a discount factor \(\beta\) such that \(q^*_j = \beta \mathbb{E}_Q [r_j]\) for each asset \(j\). Moreover, assuming an interior solution, an equilibrium is Pareto efficient if and only if there is a unique such measure \(Q\).

Proof. In equilibrium, each consumer must be optimizing, and the existence of measure \(Q\) follows immediately from (4.4) or (4.5). From Proposition 4.3, an equilibrium is efficient if and only if all of these measures coincide one another, i.e., such a measure is unique (see Remark 4.1).

This result shows in fact that an equilibrium must be efficient if the rank of return matrix \(R(p^*)\) is full (equal to \(S\)).

Proposition 5.6 If the rank of return matrix \(R(p^*)\) is full at a sequential competitive equilibrium, then the equilibrium allocation is Pareto efficient.
Proof. If \( R(p^*) \) is invertible, then from (4.2), all the \((\lambda_i^0, \lambda_i^1, \ldots, \lambda_i^S)\) must be collinear, and so the result holds by Lemma 5.5.

So when the payoff matrix is invertible, the efficiency obtains just as for an AD equilibrium. This is not a coincidence, since in fact when the matrix is invertible, an equilibrium in GEI is an AD equilibrium, and vice versa. And in this sense we can view the AD model as a benchmark.

**Proposition 5.7 (Arrow)** If payoff matrix \( R(p) \) is full rank (i.e., rank \( S \)), and there is no endowments of assets, the AD equilibrium is equivalent to the AR sequential markets equilibrium in the following sense: if \( x^* \) is an equilibrium consumption allocation for an AD equilibrium, then there are price vectors \( p^*, q^* \) and portfolio allocation \( b^* \) such that \((p^*, q^*, x^*, b^*)\) is a GEI equilibrium; conversely, if \( x^* \) is an equilibrium consumption allocation for a GEI equilibrium \((p^*, q^*, x^*, b^*)\) where \( R(p^*) \) is full rank, then there is a price vector \( p^{**} \) such that \((p^{**}, x^*)\) is an AD equilibrium.

**Proof.** (1) Suppose that \((p^*, q^*, x^*, b^*)\) is a GEI competitive equilibrium where \( R(p^*) \) has rank \( S \). Then no free lunch condition must be satisfied and so there exist state prices \( \lambda \gg 0 \) such that \( q^* = R(p^*)^T \lambda \) by Proposition 3.23. Set \( \hat{p}^* = \lambda^* p^{**} \) for \( s = 1, \ldots, S \) and \( \hat{p}^0 = p^{**} \). We claim \((\hat{p}, x^*)\) constitutes an AD equilibrium. The market clearing condition is satisfied by construction and so it suffices to verify that each consumer is utility maximizing. As we have seen in Lemma 3.25, the budget constraint for consumer \( i \) is equivalent to the auxiliary budget constraint (3.6). If \( R(p^*) \) has rank \( S \), the second condition in (3.6) is redundant, and the first constraint is nothing but the AD budget constraint. Hence \( x^*_i \) is budget feasible and utility maximizing.

(2) Let \((p^*, x^*)\) be an AD equilibrium, and suppose that \( R(p^*) \) has rank \( S \). By Lemma 3.26 we may as assume that \( R(p^*) = I \). Set \( q_j^* = 1 \) for all \( j = 1, \ldots, S \). For each \( i \), set \( b_{is}^* = p^{**} \cdot (x_i^{**} - e_i^s) \) for \( s = 1, \ldots, S \). We shall show that \((p^*, q^*, x^*, b^*)\) is an equilibrium in the sequential setting. The market clearing for the goods are trivially satisfied. The asset markets clear, since \( \sum_i b_{is}^* = p^{**} \cdot (x_i^{**} - e_i^s) = p^{**} \cdot \sum_i (x_i^{**} - e_i^s) = 0 \) by the market clearing for goods. Note that for any \((x_i, b_i)\) which satisfies the sequential budget (recall \( q^* = 1 \)), \( \sum_s p^{**} \cdot (x_i^s - e_i^s) = p^{**} \cdot (x_i^0 - e_i^0) + \sum_s p^{**} \cdot (x_i^s - e_i^s) = p^{**} \cdot (x_i^0 - e_i^0) + \sum_s b_{is} \), where the last term is zero by the budget constraint for \( s = 0 \). So it also satisfies the AD budget. We are done therefore if \((x_i^*, b_i^*)\) satisfies the sequential budget, but this can be readily confirmed by following the equation above. ■

**Remark 5.8** The asset appeared in the second half of the proof above - a special security which pays $1 in one specified state \( s \) and nothing otherwise - is often referred to as the **Arrow security** for state \( s \). Notice that we could apply Lemma 3.25 without referring to Arrow securities (it is left as an exercise).

With this result, the following definition makes sense:

**Definition 5.9** We say that the markets are complete (in equilibrium) if the return matrix \( R \) has full rank (in the equilibrium).
Remark 5.10 The assumption of $\bar{b} = 0$ is not essential, but it was made for simplicity. After all, initially owning $\bar{\theta}_i$ is equivalent to the model where consumer $i$ is endowed with $e_i + \mathbf{R}\bar{b}_i$.

Example 5.11 Elaborate on the remark above, and establish Proposition without assuming $\bar{\theta} = 0$.

5.3 Single good economy with real assets

When there is only one good in each state, and all the assets are real, the model simplifies very much. In this case, we can set $p^s = 1$ without loss of generality by the homogeneity of a competitive equilibrium (Lemma 3.18), and so consumer’s problem (3.1) is simply

$$\max_{x_i, b_i} U_i(x^0_i, x^1_i, ..., x^S_i)$$

subject to

$$x^0_i + \sum_j q_j b_{ij} = e^0_i$$

$$x^s_i - e^s_i = \sum_j b_{ij} r^s_j \text{ for each } s = 1, ..., S.$$ 

Now write $V_i(x^0_i, b_i) := U_i(x^0_i, ..., e^s_i + \sum_j b_{ij} r^s_j, ...)$. Eliminating $x$, the problem above is equivalent to:

$$\max_{x^0_i, b_i} V_i(x^0_i, b_i)$$

subject to

$$x^0_i + q \cdot b_i = e^0_i + q \cdot b_i$$

where of course $q = (q^1, ..., q^J)$ and $b_i = (b^1_i, ..., b^J_i)$, etc. Then this is a GE model of $J + 1$ goods (or equivalently, an AD model of $J + 1$ contingent goods), where good 0 is the same as the original, and $j$th “good” is nothing but the asset $j$.

Exercise 5.12 Verify that a competitive equilibrium of the $J + 1$ goods model above is a GEI equilibrium, and vice versa.
Chapter 6

Genericity Analysis and Regular Economies

When markets are incomplete, that is the rank of return matrix is less than $S$, then one should not expect that a rational expectation equilibrium is Pareto efficient (with respect to $\mathbb{R}_+^L$) in “general”. That is, if the consumption good could be re-allocated, then everybody would be made better off.

But what do we exactly mean by “in general”? The purpose of this chapter is to develop a technical tool to formalize the idea. We shall then argue that the number of Arrow - Debreu equilibria is finite in general explicitly stating its proper mathematical meaning. Intuition can easily be obtained for the case of $L = 2$, but one needs some advanced technical tools to state it generally. Throughout, I assume that the endowments are strictly positive and the consumption set $X_i = \mathbb{R}_+^L$ for each consumer.

6.1 Differentiability of demand function

Let us begin by establishing the differentiability of demand functions. The basic mathematical tool for this is the implicitly function theorem:

**Theorem 6.1 (Implicit Function Theorem)** Let $U$ be an open set in $\mathbb{R}^m \times \mathbb{R}^n$ and $f : U \rightarrow \mathbb{R}^m$ is $C^r$. Suppose that $D_x f$ is invertible at $(\bar{x}, \bar{y}) \in U \subset \mathbb{R}^m \times \mathbb{R}^n$. Then there is a unique $C^r$ function $\phi$ from a neighborhood $V$ of $\bar{y}$ to $\mathbb{R}^m$ such that $f(\phi(y), y) = f(\bar{x}, \bar{y})$ for all $y \in V$. Moreover, $D\phi(y) = -[D_x f(x, y)]^{-1} D_y f(x, y)$.

First let us establish the differentiability of the demand function in the Arrow Debreu model. Assuming that the demand occurs at an interior point, we know that the demand vector is characterized by the following first order condition:

$$DU_i(x) - \lambda p = 0$$
$$-p \cdot x + m = 0$$

Writing the left hand side as $\Phi(x, \lambda; p, m)$, we view this as a $C^1$ function on $\mathbb{R}_+^{L+1} \times \mathbb{R}_+^{L+1}$ to $\mathbb{R}_+^{L+1}$. The implicit function theorem says that $x$ and $\lambda$ are $C^1$ function

---

1In general, this is a $C^{r-1}$ function when $U_i$ is a $C^{r}$ function.
of prices \( p \) and income \( m \) if \( D_{(x, \lambda)} \Phi (x, \lambda; p, m) \) is invertible, where \( (x, \lambda; p, m) \) solves the FOC. Now \( D_{(x, \lambda)} \Phi (x, \lambda; p, m) = \begin{bmatrix} D^2 U_i & -p^T \\ -p & 0 \end{bmatrix} \), and so utilizing the first order condition, this matrix is invertible if and only if the bordered Hessian matrix

\[
\begin{bmatrix}
D^2 U_i & [DU_i]^T \\
DU_i & 0
\end{bmatrix}
\] (6.1)

is invertible. So we say a strictly quasi concave function \( U_i \) is differentiably strictly quasi concave at \( x \) if (6.1) is invertible at \( x \).

**Proposition 6.2** The Arrow Debreu demand function is \( C^1 \) if the matrix (6.1) is invertible. The derivatives of the demand function as well as the Lagrange multiplier is given by

\[
\frac{\partial x}{\partial p} \frac{\partial x}{\partial m} \frac{\partial x}{\partial \lambda} \frac{\partial \lambda}{\partial p} \frac{\partial \lambda}{\partial m} = \begin{bmatrix}
D^2 U_i & -\frac{1}{\lambda} [DU_i]^T \\
-\frac{1}{\lambda} DU_i & 0
\end{bmatrix}^{-1} \begin{bmatrix}
\lambda & 0 \\
x^T & -1
\end{bmatrix} (6.2)
\]

We always assume that \( U_i \) is strictly quasi concave, so in particular \( D^2 U_i \) is negative semi-definite. This is not enough to guarantee that the matrix above is invertible in general. So we say a function is differentiably strictly quasi-concave if it is strictly quasi concave and the matrix above is invertible.

**Exercise 6.3** Show that \( U \) is differentiably strictly quasi concave if \( DU \) is negative definite.

**Exercise 6.4** The Hicksian demand (or compensated demand) function is defined as a (unique) solution to \( \min_x p \cdot x \) subject to \( U(x) \geq \bar{u} \). Write the Hicksian demand as \( h(p, \bar{u}) \). Show that the compensated demand function is differentiable with respect to \( p \) when \( U \) is differentiably strictly quasi-concave. Also establish the Slutsky equation

\[
\frac{\partial}{\partial p} h(p, \bar{u}) = \frac{\partial}{\partial p} x(p, m) + \frac{\partial}{\partial m} x(p, m) (x(p, m))^T (6.3)
\]

where \( m = p \cdot h(p, \bar{u}) \) from (6.2) and an analogous relation for the Hicksian demand.

Now let us consider demand functions in the GEI setup. We know that the demand for goods and assets are given by the first order conditions (6.4). Recall that the first order conditions imply the no free lunch condition, that is, the demand functions are not well defined if prices violate the no free lunch condition. Hence the differentiability below should be understood as such.

We shall apply the implicit function technique as before. Recall that the first order condition for the utility maximization is the following set of equations:

\[
\begin{align*}
\frac{\partial U_i}{\partial x_i} - \lambda_i^o p^s &= 0, s = 0, 1, ..., S \\
p^0 \cdot (x_i^0 - e_i^0) + q \cdot b_i &= 0 \\
p^s \cdot (x_i^s - e_i^s) &= r^s (p^s) b_i, s = 0, 1, ..., S \\
-\lambda_i^0 q + (R(p))^T \begin{bmatrix}
\lambda_i^s \\
\vdots
\end{bmatrix} &= 0
\end{align*}
\] (6.4)
Denoting by $\Phi (x_i, \lambda_i, b_i, p, q)$ the left hand side of the first order condition, we shall apply the implicit function theorem on this: it suffices to show that the matrix of derivatives by the endogenous variables is invertible. So differentiating, by endogenous variables $x$, $\lambda$'s, and $b$, we obtain the following matrix (we will omit subscript $i$):

$$
\begin{bmatrix}
D^2U & -P & 0 \\
-P^T & 0 & R \\
0 & R^T & 0
\end{bmatrix},
$$

where $\Lambda_i$ is an $S - 1$ dimensional diagonal matrix of the multipliers, i.e.,

$$
\Lambda_i = \begin{bmatrix}
\lambda_i^0 & 0 \\
\lambda_i^1 & \ddots \\
0 & \lambda_i^S
\end{bmatrix},
$$

and $P$ is the $L \times (S + 1)$ matrix consisting of spot prices as follows:

$$
P \equiv \begin{pmatrix}
p^0 & 0 & \ldots & 0 \\
0 & p^1 & 0 & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & p^S
\end{pmatrix},
$$

and finally $R$ is is given by,

$$
R \equiv \begin{pmatrix}
-q \\
R(p)
\end{pmatrix}
$$

is a $(S + 1) \times J$ matrix (asset prices $q$ is a row vector of $J$ dimension.)

We show that this is invertible if $D^2U$ is negative definite:

\begin{proposition}
Assume that $R(p)$ has rank $J$, i.e., there is no redundant asset. Then the demand functions for goods and assets are $C^1$ functions at prices consistent with no free lunch condition if $D^2U$ is negative definite everywhere.
\end{proposition}

\textbf{Proof.} By the implicit function theorem, it suffices to show that the matrix above in invertible. So let $(\Delta x^T, \Delta \lambda^T, \Delta b^T) \in \mathbb{R}^L \times \mathbb{R}^{S+1} \times \mathbb{R}^J$ be a vector and suppose that

$$
\begin{bmatrix}
D^2U & -P & 0 \\
-P^T & 0 & R \\
0 & R^T & 0
\end{bmatrix} \begin{bmatrix}
\Delta x \\
\Delta \lambda \\
\Delta b
\end{bmatrix} = 0, \text{ i.e.,}
$$

$$
D^2U \Delta x - P \Delta \lambda = 0 \\
-P^T \Delta x + R \Delta b = 0 \\
R^T \Delta \lambda = 0
$$

\textsuperscript{2}This can be relaxed - a condition resembling the differentiably strictly quasi concavity can be stated formally.
Multiplying the second equation by $\Delta\lambda^T$ from the left, we have $-\Delta\lambda^T P^T \Delta x + \Delta\lambda^T R \Delta b = 0$. The last equation then implies that $\Delta\lambda^T P^T \Delta x = 0$. Multiplying the first equation by $\Delta x^T$ from the left, $\Delta x^T D^2 U \Delta x - \Delta x^T P \Delta \lambda = 0$. Thus $\Delta x^T D^2 U \Delta x = 0$ follows from $\Delta\lambda^T P^T \Delta x = 0$. Since $D^2 U$ is negative definite, $\Delta x = 0$ must follow. Then the first equation implies that $\Delta\lambda = 0$ and the second equation yields $R \Delta b = 0$. Since $R$ has full rank, this implies that $\Delta b = 0$. So we have shown that $(\Delta x^T, \Delta\lambda^T, \Delta b^T) = 0$, hence the matrix in question is invertible.

The implicit function theorem also tells us that the derivatives of the demand functions can be found, as we have seen for the case of the Arrow Debreu demand. The expression however gets complicated when $R(p)$ changes as $p$ change, which is often not very useful object for our analysis. But the expression is simpler in the following two scenarios, which we shall study later in detail:

1. all assets are nominal
2. all assets pays in a designated numéraire: without loss of generality, say that $L'$ th good is the numéraire whose price is always set to be one. So when spot prices other than the numéraire change, $R$ does not change. (this case include the case of one good as a special case.

Exercise 6.6 Applying the implicit function theorem, write the derivatives of $x$, $b$, and $\lambda$ analogously to the case of Arrow Debreu demand in Proposition 6.2.

### 6.2 Regular Economies - Complete market case

Let $x_i(p, m)$ be a unique solution to the consumer problem $\max_{x_i \in X_i} U_i(x_i)$ subject to $p \cdot x_i = m$. The substitution matrix is defined as:

$$
\bar{S}_i := \frac{\partial}{\partial p} x_i(p, p \cdot e_i) + \frac{\partial}{\partial m} x_i(p, p \cdot e_i)(x_i(p, p \cdot e_i) - e_i)^T,
$$

where $\frac{\partial}{\partial m} x_i(p, p \cdot e_i)$ is a column vector of income effects. From the standard duality theory, we have the following property:

**Lemma 6.7** The substitution matrix $\bar{S}_i$ is $L \times L$ is a negative semi-definite symmetric matrix and $p \cdot \bar{S}_i = 0$, Moreover, if the utility function is differentially strictly concave, it has rank $L - 1$.

Notice that the rank condition above together with $p \cdot \bar{S}_i = 0$, $p >> 0$, implies that any $L - 1 \times L - 1$ submatrix obtained by deleting $l$th row and column is negative definite.

Let $Z_i(p; e_i) := x_i(p, p \cdot e_i) - e_i$ and then $Z(p; e) := \sum_{i=1}^{I} Z_i(p; e_i) = \sum_{i=1}^{I} (x_i(p, p \cdot e_i) - e_i)$ is the aggregate excess demand function, where $e = (e_1, ..., e_I) \in (\mathbb{R}^L)^I$ is the profile of initial endowments. By homogeneity we can normalize $p^L = 1$ in order to search equilibria, so write $\hat{p}$ for the prices of the first $L - 1$ goods. Denote by $\hat{Z}(p; e)$ the
$L-1$ dimensional vector consisting of the excess demand for the first $L-1$ goods, and setting $p^L = 1$. A similar convention is used for the demand function $x_i$. By Walras law, $p = (\hat{p}, 1)$ is a vector of AD equilibrium prices if and only if $\hat{Z}(\hat{p}; e) = 0$. Also under the standard boundary condition, we have $||x_i((\hat{p}^n, 1), m^n)|| \to \infty$ as $n \to \infty$ for any sequence $(p^n, m^n)_{n=1}^{\infty}$ such that $||p^n|| \to \infty$ as $n \to \infty$ and $m^n \in [m, \bar{m}]$ for every $n$ where $m$ and $\bar{m}$ are positive constants.

Notice also that by the by construction, for any endowments $e_i$ with $p \cdot e_i = m$, $x_i(p, p \cdot e_i) = x_i(p, m)$. So in particular, for any (small) vector $\alpha \in \mathbb{R}^{L-1}$, if we set $e_i(\alpha) := e_i - [\alpha - \hat{p} \cdot \alpha] \in \mathbb{R}^L$ (6.6)

then since $p \cdot e_i(\alpha) = p \cdot e_i$, we observe that $x_i(p, p \cdot e_i(\alpha))$ is invariant of $\alpha$, and so

\[
\hat{Z}_i(p; e_i(\alpha)) = \hat{x}_i(p, p \cdot e_i(\alpha)) - \hat{e}_i(\alpha) = \hat{Z}_i(p; e_i) + \alpha
\] (6.7)

First we shall show that the set of (normalized) equilibrium prices must be compact.

**Lemma 6.8** [compactness lemma] For any compact set $K \subset (\mathbb{R}^{L+})^I$, the set $\{(\hat{p}, e) : \hat{Z}(\hat{p}; e) = 0 \text{ and } e \in K\} \subset \mathbb{R}^L \times K$ is compact.

**Proof.** The set in question is closed by the continuity of $\hat{Z}$ and the compactness of $K$. So we need to show it is bounded. Suppose that there is a sequence $(\hat{p}^n, e^n)$ such that $\hat{Z}(\hat{p}^n; e^n) = 0$ and $e^n \in K$ for every $n$, and that $|\hat{p}^n| \to \infty$. Since $K$ is compact, we can find $\bar{e} \in \mathbb{R}^L$ such that $\sum_{i=1}^{I} e_i^n \leq \bar{e}$ for all $n$. But then $0 = Z(p^n; e^n) \geq \sum_{i=1}^{I} x_i(p^n, p^n \cdot e_i^n) - \bar{e}$ must hold for any $n$ which contradicts the boundary condition for the demand functions.

Fix utility functions throughout, so that the economies are parametrized by $e$. Thus we shall refer to $e$ as an economy in the following.

**Definition 6.9** An equilibrium $p$ of economy $e$ is said to be **regular** if $D_p \hat{Z}(p; e)$ has rank $L - 1$. An equilibrium said to be critical if it is not regular. An economy $e$ is a **regular economy** if every equilibrium is regular. An economy which is not regular is said to be a singular economy.

First off all, there are trivial regular economies as shown below.

**Lemma 6.10** If economy $e$ is efficient (i.e., the initial endowments are efficient), then $x = e$ is a unique equilibrium and it is regular. Thus such an economy is regular.

**Proof.** Notice that by definition, at an equilibrium allocation every consumer must be at least as well off as at the initially endowed bundle of goods, so if $e$ is efficient, then this must be a unique equilibrium since we are assuming strict convexity of preferences. Denote by $\hat{p}^*$ the unique price system supporting the equilibrium $x = e$. 

39
To see the regularity, from (6.5), for every $i$, $\bar{S}_i = \frac{\partial}{\partial p} x_i(p, p \cdot e_i)$ is a negative semi-definite symmetric matrix of rank $L - 1$ and $p^* \cdot \bar{S}_i = 0$, since $x_i(p, p \cdot e_i) - e_i^T = 0$.

Thus $\frac{\partial}{\partial p} Z(p^*, e) = \sum_{i=1}^{I} \bar{S}_i$ is a negative semi-definite symmetric matrix of rank $L - 1$ and $p^* \cdot \sum_{i=1}^{I} \bar{S}_i = 0$. Since $p^* \gg 0$, this means that $D_p Z(p^*, e)$ must be of rank $L - 1$, since it is the submatrix of $\sum_{i=1}^{I} \bar{S}_i$ where the last row and the last column are deleted.

An immediate implication of Lemma 6.10 is that if the model admits a representative agent (i.e., effectively there is one consumer) then the determinacy issue is trivial.

**Proposition 6.11** A regular economy $e^*$ has finitely many equilibria.

**Proof.** This is an implication of the implicit function theorem and the boundary property of the demand function. Notice that the regularity of an equilibrium $p^*$ means that $\hat{Z}(\hat{p}; e) = 0$ has a locally unique (differentiable) solution $\hat{p}(e)$ around $(p^*, e^*)$. This means that by Lemma 6.8, a limit point of any sequence of equilibrium prices must be an isolated equilibrium price system. Hence the number of equilibria must be finite. ■

### 6.2.1 Genericity of regular economies

We shall now argue that “most” economies are regular. It is an application of Sard’s theorem:

**Theorem 6.12 (Sard’s Theorem)** Let $f$ be a $C^k$ function from an open set $V$ in $\mathbb{R}^n$ to $\mathbb{R}^m$, where $k > \max\{n - m, 0\}$. Then the set of critical values of $f$, $$\{f(v) : \text{rank} Df(v) < m, v \in V\},$$ has Lebesgue measure 0 in $\mathbb{R}^m$.

Sard’s theorem is a very deep result and we do not provide a proof here. In applications, the following variant of Sard’s theorem is useful, which can be shown by Sard’s theorem:

**Theorem 6.13 (Transversality Theorem (simple form))** Let $f$ be a $C^k$ function from $V \times W$ to $\mathbb{R}^m$, where $V$ and $W$ are open sets in $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively, and $k > \max\{n, 0\}$. Suppose 0 is a regular value of $f$, i.e., $\text{rank} Df(v, w) = m$ whenever $f(v, w) = 0$. Then the set $\{w \in W : 0$ is a critical value of $f(\cdot, w)\}$ has Lebesgue measure zero.

**Proof.** Let $X := \{(v, w) : f(v, w) = 0\}$ and consider the natural projection $\pi(v, w) = w$ from $X$ to $\mathbb{R}^l$. By Sard’s theorem the critical value of $\pi$ is a measure zero set. Then it suffices to show that $w$ is a critical value of $\pi$ if and only if 0 is a critical value of $f(\cdot, w)$. Notice that 0 is a critical value of $f(\cdot, w)$ if and only if $w$ is a regular value of $\pi$. ■

The transversality theorem is very powerful and useful in two ways. If $m = n$, then it says we can apply the implicit function theorem to $f(v, w) = 0$ to solve $v$ in $w$. If $m > n$, then $\frac{\partial}{\partial v} f$ can never be invertible, so the theorem says that $f(\cdot, w) = 0$ cannot have a solution almost everywhere in $w$. 40
Example 6.14 Let \( f(x_1, x_2, a_1, a_2) = \begin{pmatrix} a_1 x_1 - 1 \\ a_2 x_2 - 1 \end{pmatrix} \), then we have \( Df(x_1, x_2, a_1, a_2) = \begin{pmatrix} a_1 & 0 & x_1 & 0 \\ 0 & a_2 & 0 & x_2 \end{pmatrix} \). As a function on \( \mathbb{R}^2 \times \mathbb{R}^2 \), 0 is a regular value of \( f \); \( f(x_1, x_2, a_1, a_2) = 0 \) implies that \( x_1, x_2, a_1, a_2 \) are all non zero and so \( Df(x_1, x_2, a_1, a_2) \) has rank 2. So by the transversality theorem 0 is a regular value of \( f(\cdot, \cdot, a_1, a_2) \) for almost all \((a_1, a_2)\). Indeed, as long as \( a_1 a_2 \neq 0 \), \( \partial_{\dot{x}} f(x_1, x_2, a_1, a_2) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \) has rank 2. Now if we set \( V = \mathbb{R} \) and \( W = \{ (x_2, a_1, a_2) \in \mathbb{R}^3 \}, \) the transversality theorem says that for almost all \((x_1, a_1, a_2) \in W, 0 \) must be a regular value of \( f(\cdot, x_2, a_1, a_2) \). Indeed, as long as \( a_2 x_2 \neq 1 \), \( f(x_1, x_2, a_1, a_2) = 0 \) never occurs, and hence 0 is a regular value trivially.

Proposition 6.15 The set of singular economies constitutes is a closed set with Lebesgue measure 0 in \( (\mathbb{R}^4)_{++} \). Thus in particular, the set of regular economies is open and its complement is a Lebesgue measure zero set of \( (\mathbb{R}^4)_{++} \).

Proof. The openness part follows from Lemma 6.8 and the continuity. For the measure statement we apply the transversality theorem: it suffices to show that for any fixed \( \hat{p} \) the set of singular economies such that \( \hat{Z}(\hat{p}; e) = 0 \) is very thin. But notice that once we fix \( \hat{p} \), as long as \( \hat{Z}(\hat{p}; e) = 0 \) is satisfied, the determinant is a polynomial function of \( e \), and we know it is non zero since there is at least one regular economy \( e^* \) where \( \hat{p} \) is the unique equilibrium prices (by Proposition 3.7 and Lemma 6.10). So on the hyperplane \( \{ e : p^* \cdot (e^* - e) = 0 \} \) the solution to \( \det D_{\hat{p}} \hat{Z}(\hat{p}; e) = 0 \) consists of the zeros of non-trivial polynomial function. The set of all singular economies is obtained by taking the union of these solutions for different \( \hat{p} \), which must be very thin.

6.2.2 The number of equilibria and its existence

The index of equilibrium \( \hat{p} \) is defined by

\[
I(\hat{p}; e) := -\text{sign} \left( \det D_{\hat{p}} \hat{Z}(\hat{p}) \right)
\]

From the Poincaré-Hopf theorem, it can be shown that:

Proposition 6.16 If \( e \) is a regular economy, then the sum of the indices for all the equilibria is one.
For a generic economy, we know that the index of an economy must be +1. Thus if one could show that at any equilibrium, the index associated with the equilibrium is positive, then there could only be one equilibrium. To put it differently, we can conclude that there must be multiple equilibria if we can find an equilibrium with negative index.

Exercise 6.17 Consider an exchange economy with two goods. The consumers characteristics are summarized below: Consumer 1: utility function \( u(x_1) + y_1 \), endowed with \((1 - \alpha), \alpha)\), Consumer 2: utility function \( x_2 + u(y_2) \), endowed with \((\alpha, (1 - \alpha))\), where \( \alpha \in (0, \frac{1}{2}) \). Show that there is a unique equilibrium where each consumer consumes \( \left( \frac{1}{2}, \frac{1}{2} \right) \) whose index is one if \( \alpha = \frac{1}{2} \). Show that this consumption allocation is an equilibrium for any \( \alpha \), but its index becomes \(-1\) if \( \alpha \) is small enough.

6.3 Regular Economies for GEI model

The regularity analysis can be developed for the GEI model, but there is one difficulty: the span of assets may depend on the prices and so we do not know a priori if some assets are redundant in equilibrium - recall that the differentability result for the demand function requires non-existence of redundant assets.

So here we develop the theory for the case where the asset span does not depend on spot prices. From now on, assume that all the assets are real, and they all pay in good 1 in every state (including \( s = 0 \)). Because of homogeneity we may normalize spot price in each state, and it is then natural to normalize the price of good 1 equal to one in every state. That is, the good 1 is the numéraire of this economy and the returns of the assets are pre-determined in units of numéraire - this kind of assets are commonly referred to as the real numéraire assets.

Throughout, assume that \( J \leq S \) and the rank of the return matrix \( R \) is \( J \), i.e., it has full rank. By Proposition 6.5, we know that the demand functions are \( C^1 \). So denote by \( x_i(p, q; e_i) \in \mathbb{R}^L \) the demand function for goods and \( b_i(p, q; e_i) \in \mathbb{R}^J \) the demand function for assets. Let \( Z_G(p, q; e) := \sum_{i=1}^I (x_i(p, q; e_i) - e_i) \) and \( Z_A(p, q; e) = \sum_{i=1}^I b_i(p, q; e_i) \). Set \( Z(p, q; e) := \begin{bmatrix} Z_G(p, q; e) \\ Z_A(p, q; e) \end{bmatrix} \in \mathbb{R}^L \times \mathbb{R}^J \). Then \((p, q)\) constitute an equilibrium if and only if \( \hat{Z}(p, q; e) = 0 \). Define as before \( \hat{Z}_G(p, q; e) \) the demand for non-numéraire goods, and set \( \hat{Z}_G(p, q; e) = \begin{bmatrix} Z_G(p, q; e) \\ Z_A(p, q; e) \end{bmatrix} \in \mathbb{R}^L \times \mathbb{R}^J \). Walras law tells us that \( \hat{Z}(p, q; e) = 0 \) holds if and only if \((p, q)\) constitute an equilibrium for economy \( e \).

First we start with the compactness lemma:

**Lemma 6.18** [compactness lemma for real numéraire economies] For any compact set \( K \subset (\mathbb{R}_+^L \times \mathbb{R}^J)^I \), the set \( \{(\hat{p}, q, e) : \hat{Z}(\hat{p}, q; e) = 0 \text{ and } e \in K\} \subset \mathbb{R}_+^{L-(S+1)} \times \mathbb{R}^J \times K \) is compact.

**Proof.** The set in question is obviously closed, so we shall argue that it is included in some compact set. Since the consumption sets are bounded below, the set of
consumption bundles which is no worse than the initial endowments are bounded from below. So the set \( \Xi := \{ x \in (\mathbb{R}^L)^J : \text{for some } e \in K, U_i(x_i) \geq U_i(e_i) \text{ for all } i \text{ and } \sum_{i=1}^J (x_i - e_i) = 0 \} \) is a compact set. From the first order condition of utility maximization, the spot prices must be proportional to the gradients, and hence the equilibrium spot prices must be included in \( \{ \overline{DU}_1(x_1) : \exists x \in \Xi, x_1 = x_1 \} \) where \( \overline{DU}_1 \) is the gradient vector normalized spot by spot, dividing with the marginal utility of the numéraire. The function \( \overline{DU}_1 \) is well defined and continuous on the set \( \{ x_i : U_i(x_i) \geq U_i(e_i) \} \), hence \( \{ \overline{DU}_1(x_1) : \exists x \in \Xi, x_1 = x_1 \} \) must be compact as the continuous image of a compact set. Notice that by the no arbitrage condition and the normalization of spot prices, equilibrium asset prices obtains by taking the marginal utility of numéraire as spot prices. So they must also be in a compact set.

Let us count the number of equations and unknowns in the system of equations which characterize the equilibrium. There are \( L \) markets for goods and \( J \) markets for assets, but by Walras’ law which applies state by state, there are \( S - 1 \) redundant equations, so there are effectively \( L + J - (S - 1) \) equations. For the prices, we have already normalized price of good 1 in every state so there are \( L - (S - 1) \) prices for goods, and there are \( J \) prices for assets, so the total number of prices is \( L + J - (S - 1) \). After all, we have the same number of equations and prices, just like the case of the complete markets. So we have the following natural definition of a regular economy.

**Definition 6.19** A GEI equilibrium is said to be regular if the rank of \( \frac{\partial}{\partial (p,q)} Z(p,q;e) \) is \( L + J - (S - 1) \). An economy \( e \) is regular if any equilibrium of it is regular.

**Lemma 6.20** 0 is a regular value for \( \hat{Z} \).

**Proof.** Let \((p, q)\) satisfy \( \hat{Z}(p, q : e) = 0 \). Basically, it suffices to show that if we look at proper columns of the matrix \( \frac{\partial}{\partial e} \hat{Z}(p, q; e) = \left[ \frac{\partial}{\partial e} \hat{Z}_G(p, q; e) \quad \frac{\partial}{\partial e} Z_A(p, q; e) \right] \), then it has the following shape:

\[
\begin{bmatrix}
\text{full rank} & * \\
0 & \text{full rank}
\end{bmatrix}
\]

That is, there are some directions of changes in \( e \) which induces any direction of changes in excess demand for goods, keeping the asset excess demand unchanged, and also there are some directions of changes in \( e \) which induces any direction of changes in excess demand for assets. This is what we shall demonstrate below.

If the perturbation argument for the complete market case (see (6.6) and (6.7)) is applied to each spot, i.e., change endowments so that the income in spot \( s \) remains constant, notice that the excess demand vector for (non-numéraire) goods changes in any direction but the asset demand remains the same. This takes care of the first set of changes. So it suffices to demonstrate that there exists a perturbation of endowments so that the demand for asset \( j \) changes but nothing else changes. Consider \( e_s^i(\alpha) := e_s^i - \left[ \alpha \begin{array}{c} \nu_s^j \\ 0 \end{array} \right] \in \mathbb{R}^{L_s}, \ s = 1, ..., S, \) and \( e_s^0(\alpha) := e_s^0 + q^j \alpha. \) Then demand \( b_s^j \) for asset \( j \) increases exactly \( \alpha \) and nothing else changes. This can be
confirmed from the FOC of utility maximization: in (6.4), using the same $\lambda_i$, we just need to confirm that the budget constraints are satisfied, but this follows by construction. This takes care of the second set of changes.

**Proposition 6.21 (Generic Regularity of real numéraire economies)** The set of regular economies is an open and full measure set in $\mathbb{R}_+^L$.

**Proof.** The openness follows from Lemma 6.18. The full measure property follows from Lemma 6.20 and the transversality theorem.
Chapter 7

Indeterminacy of Equilibria

7.1 Generic finiteness for the case of real assets

From the compactness lemma (Lemma 6.18), it can be readily established that a regular economy has a finitely many equilibria.

Exercise 7.1 Prove that a regular economy has a finitely many equilibria.

So the generic regularity result (Proposition 6.21), we conclude that for the real numéraire economies, generically the number of equilibria is finite and the number is locally constant.

This result is very intuitive once we believe that if the number of equations and the unknowns are the same, there should be finitely many solutions to it, generically. Indeed, this is the case for the case of AD equilibria. For the case of real numéraire assets, this is the case, too: there are just as many (normalized) prices as the number of markets since Walras law holds spot by spot. The intuition suggests that the result extends for the case of real assets: again, the number of effective prices and the number of effective market clearing conditions are the same: however, it gets more complicated since we need to worry about the case where the rank of asset span suddenly changes.

7.2 Generic indeterminacy of GEI equilibria with nominal assets

Now we turn to the case of nominal assets: each asset pays in units of account. Recall that when the markets are complete, the type of assets does not matter; as long as the dimension of the asset span is $S$, we have the equivalence to the AD equilibrium (Proposition 5.7). However, this is not so when markets are incomplete.
Consumer’s problem is
\[
\max_{x_i, b_i} U_i \left( x^0_i, x^1_i, ..., x^S_i \right) \tag{7.1}
\]
subject to
\[
x^0_i + \sum_j q_j b_{ij} = e^0_i \\
p^s (x^s_i - e^s_i) = \sum_j b_{ij} r^s_j \text{ for each } s = 1, ..., S.
\]

It can be readily observed that the budget is no longer homogeneous in prices. Notice that if we set \( \gamma^s := 1/p^s \), then the problem is equivalent to the case of real assets where asset \( j \)'s return in state \( s \) is \( \gamma^s r^s_j \). In other words, once \( \gamma^s, s = 1, ..., S \) are fixed, the model is equivalent to the real asset model with return matrix \( \Gamma R \), where \( \Gamma \) is the \( S \times S \) diagonal matrix consisting whose diagonal elements are \( \gamma^s, s = 1, ..., S \):

\[
\Gamma := \begin{pmatrix}
\cdots & 0 \\
\gamma^s \\
0 & \cdots
\end{pmatrix}
\]

Proceeding as in the case of real assets, write \( \phi (\pi, \Gamma, e) \) for the auxiliary excess demand function where asset returns structure is given by \( \Gamma R \). Pick a regular economy \( e \), then by the implicit function theorem, each equilibrium \( \pi \) can be written as a smooth function of \( S \) dimensional vector \( \gamma = (\cdots, \gamma^s, \cdots) \).

So there will be indeterminacy - there will be a continuum of equilibria, parametrized by \( \gamma \). An interesting question is to ask dimensionality of the set of equilibria.

It is clear that the dimension must be at most \( S - 1 \), since for any positive number \( t \), the set of equilibria for \( t\Gamma R \) is the same as that for \( \Gamma R \): simply the asset prices are multiplied by \( t \) and nothing else will change; i.e., we loose at least one degree of freedom because of this property.

A conservative estimate is as follows. Suppose that we change \( \gamma \) in such a way that induced asset prices \( \pi (\gamma) \Gamma R \) are kept constant, which constitutes \( J \) constraints. Then since the shape of no-arbitrage condition is maintained, it is possible this change is absorbed by nominal changes. But otherwise the equilibrium should change - so there will be at least \( S - 1 - J \) dimensional indeterminacy.

But in general, the degree of indeterminacy should be larger, and in fact if the number of consumers is larger than the number of assets, then the degree of indeterminacy will be \( S - 1 \). This is a striking difference from the case of Arrow-Debreu complete market case, where generically the number of equilibria should be finite. Moreover, it does not matter if assets are real or nominal.

I shall sketch the argument for the indeterminacy result:

**Lemma 7.2** Suppose that every \( J \times J \) submatrices of \( R \) is invertible, hence all the raw vectors of \( R \) are linearly independent. If the columns of \( \Gamma R \) and the columns of \( \Gamma' R \) span the same subspace, then there is a number \( t \) such that \( t\Gamma = \Gamma' \).
Proof. If the columns of $\Gamma R$ and the columns of $\Gamma' R$ span the same subspace, there must be $J \times J$ invertible matrix $M$ such that $(\Gamma')^{-1} \Gamma R = RM$. It means that every row vector of $R$ is eigen vector of $M$, corresponding eigen values are the diagonal elements of $(\Gamma')^{-1} \Gamma$. But recall that the eigen vectors constitute a direct sum of linear subspaces that spans $\mathbb{R}^J$: that is, $\mathbb{R}^J = V_1 \oplus V_2 \oplus \cdots \oplus V_N$ where each subspace $V_n$ corresponds to a distinct eigen value. Since the row vectors are linearly independent so that the dimension of each $V_n$ in the direct sum is the same as the number of the row vectors of $R$ it contains or else it has dimension $J$. Since $S > J$, this is possible only when $N = 1$. Therefore, the diagonal elements of $(\Gamma')^{-1} \Gamma$ must be identical, hence $t \Gamma = \Gamma'$ for some $t$. ■

Lemma 7.3 Consider two equilibria: one for $\Gamma$ and another for $\Gamma'$. If the columns of $\Gamma R$ and the columns of $\Gamma' R$ span different subspaces and the matrices of equilibrium portfolios has rank $J$, then the equilibrium consumption vectors must be different.

The following is due to Geanakoplos and Mas-Colell (1989).\footnote{Balasko and Cass (1989) have independently established an indeterminacy result. The spirit of their approach is closer to the conservative estimate described in the text.}

Proposition 7.4 If $I > J$, then generically in $e$, the set of equilibrium consumption vectors have dimension $S - 1$. 

\footnote{Balasko and Cass (1989) have independently established an indeterminacy result. The spirit of their approach is closer to the conservative estimate described in the text.}
Chapter 8

Inefficiency and Constrained (in)Efficiency in Incomplete Markets

8.1 Generic Pareto Inefficiency

It should be no surprise that a GEI equilibrium is Pareto inefficient except for some knife edge cases.

Example 8.1 In the background risk model, an equilibrium is not efficient except for non-generic cases. Indeed, recall that the efficiency is equivalent to the condition that the measure associated with the Lagrange multipliers (See Remark 4.1 and Lemma 5.5) is identical across the consumers, whereas the equilibrium condition only says that the implied discount factors \( \beta_i \) (see Lemma 5.5) are common.

Exercise 8.2 Suppose that \( \sum_i Y_i = \bar{y} \) for sure in the background risk model. Show that, unless every \( Y_i \) is non-random, an equilibrium cannot be Pareto efficient.

Here we shall give a formal analysis for generic inefficiency. Recall that the efficiency is equivalent to the coincidence of personal state prices, which are essentially the Lagrange multipliers \( \lambda_i \) corresponding the budget in each state. So we shall see if these vectors of multipliers rarely coincide.

In chapter 6.1 we have seen that under our assumptions, the demand as well as the Lagrange multipliers \( \lambda_i \) are functions of prices as well as endowments, so write it as \( \lambda_i (p, q; e_i) \). We shall first show that for given prices, we can change endowments so that function \( \lambda_i \) moves any direction orthogonal to the span of assets while the excess demand for goods and assets remains the same. Note that if markets are complete, the orthogonal complement of the asset span is null, so this perturbation works only when markets are incomplete.

Lemma 8.3 For any \( \xi \in \mathbb{R}^S \) such that \( R(p)^T \xi = 0 \), there is a smooth path \( e_i (t) \) defined for small enough \( t \in \mathbb{R} \) with \( e_i (0) = e_i \) such that \( x_i (p, q; e_i (t)) - e (t) \) and \( b_i (p, q; e_i (t)) \) do not depend on \( t \) and \( \frac{dt}{dt} \lambda^s_i (p, q; e_i (t)) = \xi^s \) for \( s = 1, \ldots, S \).
Moreover, of constraints (3.2) are satisfied by keeping minimization: first observe that the net trade of goods is invariant of endowed with for all derivatives of the left hand side of the equations above with respect to endowed with . By construction, for every small enough. Now set . WITHOUT loss of generality, assume that . Then we can find suitable coordinates so that the second equation can be changed by . Moreover, so that . Without loss of generality, assume that . Without loss of generality, assume that . Consider the following system of equations:

\[ \begin{align*}
\dot{Z}(p, q; e) &= 0, \\
\frac{\lambda_1^1(p, q; e)}{\lambda_0^0(p, q; e)} - \frac{\lambda_1^2(p, q; e)}{\lambda_2^0(p, q; e)} &= 0.
\end{align*} \]

The second equation must hold if the multipliers are proportional to each other, i.e., the allocation is Pareto efficient. We have already seen that \( \frac{\partial}{\partial e_1} \dot{Z}(p, q; e) \) has full rank. By Lemma 8.3, the second equation can be changed by \( \xi^1 \) by perturbing \( e_2 \) without affecting \( \dot{Z}(p, q; e) \). Then we can find suitable coordinates so that the derivatives of the left hand side of the equations above with respect to \( e_1 \) and \( e_2 \) has the form

\[ \begin{bmatrix}
I & * \\
0 & \xi^1
\end{bmatrix}, \]

which has full rank. To sum up, 0 is a regular value of the function defined by the left hand side of the equations.

By the transversality theorem, for almost every choice of endowments \( e \), 0 must be a regular value of the function which assigns the left hand side to \( (p, q) \). But

\[ \text{\textsuperscript{1}If } U \text{ is strictly concave and } D^2U \text{ is negative definite at } x, \text{ then this property follows directly by the implicit function theorem applied to the simultaneous equations} \frac{\partial}{\partial x} U(x) \lambda^s(t) p^s = 0, \text{ for } s = 1, \ldots, S, \text{ around } t = 0. \]

Proof. Fix prices \( p \) and \( q \), and pick any \( \xi \in \mathbb{R}^S \) such that \( R(p)^T \xi = 0 \). For \( t \in \mathbb{R} \), let \( \lambda^s(t) = \lambda^0 + t \xi^s \) for \( s = 1, \ldots, S \) and \( \lambda^0(t) = \lambda^0 \). If we choose \( t \) small enough, solving the utility maximization problem with auxiliary budget constraint (3.6), we find the demand vector \( x_i(t) \) corresponding to the state prices \( \lambda_i(t) \): that is, \( x_i(t) \) sailishes

\[ \frac{\partial U_i}{\partial x_i} \left( x(t) \right) = \lambda^s_i(t) p^s, \text{ for } s = 0, 1, \ldots, S, \]

for all \( t \) small enough. Now set \( e_i(t) = x_i(t) - (\bar{x}_i - e_i) \).

We claim that the demand vector is exactly \( (x_i(t), \lambda_i(t), b_i) \) if consumer \( i \) is endowed with \( e_i(t) \). To see this, consider the first order condition for utility maximization: first observe that the net trade of goods is invariant of \( t \), and so the budget constraints (3.2) are satisfied by keeping \( b_i \) fixed. Next, observe that the marginal utility condition (4.1) are equated with spot prices in each spot by the construction of \( x(t) \). Finally, the no arbitrage condition (4.2) is met because \( R(p)^T \xi = 0 \).

By construction, \( x_i(p, q; e_i(t)) - e(t) = \bar{x}_i - e_i \) and \( b_i(p, q; e_i(t)) = b_i \) for any \( t \). Moreover, \( \lambda^s_i(p, q; e_i(t)) = \lambda^s_i(t) \) so \( \frac{d}{dt} \lambda^s_i(p, q; e_i(t)) = \frac{\partial}{\partial e_1} \lambda^s_i(p, q; e_i(t)) \frac{d}{dt} e_i(t) = \xi^s \) for every \( s \), as we wanted. \( \blacksquare \)

Now we are ready to prove the generic inefficiency result for real numéraire economies.

Proposition 8.4 Let \( I > 1 \) and consider numéraire assets whose asset span has rank \( J < S \). Then generically in endowments, any equilibrium of economy \( e \) is Pareto inefficient.

Proof. The openness argument again follows from the compactness lemma (Lemma 6.18). So we shall apply the transversality theorem to establish this result.

Since \( J < S \), there is a non-zero row vector \( \xi \in \mathbb{R}^S \) with \( \xi R = 0 \). Without loss of generality, assume that \( \xi^1 \neq 0 \). Consider the following system of equations:

\[ \begin{align*}
\dot{Z}(p, q; e) &= 0, \\
\frac{\lambda_1^1(p, q; e)}{\lambda_0^0(p, q; e)} - \frac{\lambda_1^2(p, q; e)}{\lambda_2^0(p, q; e)} &= 0.
\end{align*} \]

The second equation must hold if the multipliers are proportional to each other, i.e., the allocation is Pareto efficient. We have already seen that \( \frac{\partial}{\partial e_1} \dot{Z}(p, q; e) \) has full rank. By Lemma 8.3, the second equation can be changed by \( \xi^1 \) by perturbing \( e_2 \) without affecting \( \dot{Z}(p, q; e) \). Then we can find suitable coordinates so that the derivatives of the left hand side of the equations above with respect to \( e_1 \) and \( e_2 \) has the form

\[ \begin{bmatrix}
I & * \\
0 & \xi^1
\end{bmatrix}, \]

which has full rank. To sum up, 0 is a regular value of the function defined by the left hand side of the equations.

By the transversality theorem, for almost every choice of endowments \( e \), 0 must be a regular value of the function which assigns the left hand side to \( (p, q) \). But
the number of equations is one more than the number of prices because of the last equation concerning efficiency, this proves that there is no solution to the system for almost every \( e \). In particular, it means that if \((p, q)\) is an equilibrium then the last equation cannot be met. So we conclude that for almost every \( e \), any equilibrium is Pareto inefficient.

**Exercise 8.5** Prove that generically in \( e \), the personal pricing measures of consumers are all different from each other in equilibrium.

### 8.2 Understanding Source of Inefficiency - Social Nash Efficiency

The generic inefficiency result does not readily indicate that competitive markets do not function well, since it is based on somewhat unfair comparison. When markets are incomplete to begin with, the resulting inefficiency is hardly surprising since the efficiency is judged under the presumption of readily reallocated consumption goods. It is too much to hope that the incomplete markets induce completely efficient use of resources, since the efficiency question can be understood as a planner’s problem, where the planner can transfer any welfare relevant goods across the consumers. On the other hand, when the markets are incomplete, markets are constrained to a restricted set of transfers. Then a more fruitful question is to ask if the markets produce an efficient outcome given such a restriction. But obviously the stronger the constraints are, the more likely the outcome is efficient with respect to them.

In this chapter we shall ask for a reasonable, but strong enough constraints, so that an GEI equilibrium is efficient. We will then see the source of inefficiency more clearly.

An immediate consequence of the first fundamental theorem of welfare economics (Proposition 5.1) is that if only complete components of markets are active, the resulting allocation should be efficient. I.e., if we keep consumers’ activities in other spots fixed, then a reallocation confined in a particular state cannot generate a Pareto improvement. But we wish to say more, especially on the asset markets.

Throughout the analysis, we shall assume that assets are real: asset \( j \) yields a bundle of goods \( r^s_j \in \mathbb{R}^L \) in state \( s \). So if consumer holds \( b_j \) units of asset \( j \), it is equivalent to a bundle \((r^s_j b_j)^S \) of goods in the next period.

**Definition 8.6** For \( s = 1, ..., S \), a direct net transfer at state \( s \) is a profile of consumption vectors \( z = (z_1, ..., z_I) \in (\mathbb{R}^L)^I \) such that \( \sum_{i=1}^I z_i = 0 \) and for every \( i \), \( z_i^{s'} = 0 \) for \( s' \neq s \). For \( s = 0 \) (i.e., period 0), a direct net transfer at state 0 is a profile of consumption vectors \( z = (z_1, ..., z_I) \in (\mathbb{R}^L)^I \) such that \( \sum_{i=1}^I z_i = 0 \) and for every \( i \), there is a portfolio \( \Delta b_i \) of assets such that \( z_i^s = \sum_{j=1}^J r^s_j \Delta b_j \) for \( s = 1, ..., S \).

That is, a direct net transfer at state \( s \) describes a reallocation of all contingent goods which can be done directly through the markets available in state \( s \). For instance, for \( s = 0 \), since there are markets for goods in period 0, there is no
restriction for $z_i^0$, and since assets markets are available, a reallocation of future contingent goods which can be achieved directly by some trade of assets is deemed feasible.

**Definition 8.7** *(Social Nash efficiency)* An allocation of consumption goods, $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_I)$, is **social Nash efficient** if for any $s = 0, 1, \ldots, S$, there is no direct net transfer $z$ at state $s$ such that $U_i(\hat{x}_i + z_i) > U_i(\hat{x}_i)$ for every $i$.

Notice that the class of transfers considered above is limited. Also notice that there is a flavor of non-cooperative optimization in the definition above, which is why the word Nash is used; the set of transfers to check efficiency is conditional on the state $s$, so one can imagine $S + 1$ independent and non-cooperative planners who are seeking for Pareto improvement. An allocation is social Nash efficient if none of these planners can induce a Pareto improvement, given all the other planners’ actions (which is to do no transfer).

**Proposition 8.8** A GEI equilibrium allocation $x^*$ is social Nash efficient.\(^2\)

**Proof.** For $s = 1, \ldots, S$, a Pareto improvement is impossible by the first fundamental theorem of welfare economics since the spot markets are complete by assumption. For $s = 0$, suppose that there is a direct net transfer $z$ at $s = 0$ which makes every consumer better off. Recall that the no arbitrage condition holds in equilibrium and so there are state prices $\lambda >> 0$ and the budget constraint of consumer $i$ is equivalent to (3.6) by Lemma 3.25. By definition, for every $i$, there is a portfolio $\Delta b_i$ of assets such that $z^s_i = \sum_{j=1}^J r^s_j \Delta b_j$ for $s = 1, \ldots, S$, thus in particular $p^s \cdot (x^*_i + z^s_i - e^*_i) = p^s \cdot \sum_{j=1}^J r^s_j (b^*_i + \Delta b_j)$ holds for every $s$. Thus $p^{s0} \cdot (x^*_i + z^0_i - e^*_i) + \sum_{s=1}^S \lambda^s p^{ss} \cdot (x^*_i + z^s_i - e^*_i) > 0$ must hold for every $i$, or else consumer $i$ could afford $x^*_i + z_i$ which is better than $x^*_i$ by the hypothesis. But then by summing up in $i$, we have $\sum_i p^{s0} \cdot z^0_i + \sum_{s=1}^S \lambda^s p^{ss} \cdot z^s_i > 0$, contradicting the condition $\sum_{i=1}^I z^s_i = 0$ for every $s$.

A careful examination of the proof above reveals that the same argument would go through if we just require that the market value of transfer in each state is the same as the value of portfolio; that is, there is a portfolio $\Delta b_i$ of assets such that $p^{ss} \cdot z^s_i = p^{ss} \cdot \sum_{j=1}^J r^s_j \Delta b_j$ for $s = 1, \ldots, S$.

**Exercise 8.9** State the result above formally and prove it.

Although more general, this extension is somewhat uncomfortable, since the efficiency notion depends on the prevailing equilibrium prices. Nevertheless, it tells us an important lesson: incomplete markets do allocate resources efficiently in the sense of social Nash efficiency, as long as the consumers expect that the equilibrium prices of future goods are independent of transfers. But such a transfer will change demand and supply and hence must have some effects in prices. In the world of rational expectations, the assumption of fixed prices is not very convincing, not to mention the assumption that the future activities in spot markets are kept independent of the current transfer. We shall explore therefore the case where the changes in prices are taken care of in the next section.

8.3 Generic Constrained Inefficiency

We want to take into account that a transfer in one state can change the condition in the other state. The idea of social Nash by design does not take care of this. We need to stick to Pareto improvement of the entire economy, and do the so called second best analysis here.

Then what should be the natural constraints? Since the incompleteness arises due to the insufficient asset structure, it is natural to require that a consumption allocation for comparison should also be attained through the existing assets.

In the following I shall concentrate on the case of real assets, and for ease of exposition I shall also assume that there is no consumption in period 0 and the net endowments of assets are all zero. Write \( r^s = (r^s_1, ..., r^s_J) \) for the matrix of real returns of assets, where \( r^s_j \in \mathbb{R}^L' \) is the bundle of goods which asset \( j \) yields in state \( s \). So if a consumer holds a portfolio \( b \in \mathbb{R}^J \), then in effect he is endowed with a bundle of goods \( e^s_i + r^s b \in \mathbb{R}^L' \) in state \( s \). By definition, consumers will trade goods in state \( s \) spot markets taking prices as given, and all the spot markets must clear in equilibrium. The resulting allocation of consumption goods is indeed what some re-allocation of the existing assets can attain.

Definition 8.10 (Constrained feasibility and efficiency) A consumption allocation \( x \) is constrained feasible if there are prices \( p^s \), \( s = 1, ..., S \), and asset trade \((b_i)_{i=1}^I\) with \( \sum_i b_i = 0 \) such that under prices \( p^s \), \((x^s_i)_{i=1}^I\) constitutes an equilibrium of the economy where consumer \( i \) is endowed with \( e^s_i + r^s b \) at every state \( s \): that is, for each consumer at every \( s \), \( x^s_i \) is utility maximizing under the budget constraint of \( p^s \cdot x^s_i = p^s \cdot (e^s_i + r^s b) \), and \( \sum_i x^s_i = \sum_i e^s_i \) in every \( s \). A constrained feasible allocation \( x \) is said to be weakly constrained efficient if there is no constrained feasible allocation \( x' \) such that \( U_i(x'_i) > U_i(x_i) \) for every \( i \).

It is clear if \((p^*, q^*, x^*, b^*)\) is a competitive equilibrium, then \( x^* \) is constrained feasible. Thus the question we want to ask is whether or not \( x^* \) is constrained efficient.

Proposition 8.11 If there is one good in each state, then a competitive equilibrium is weakly constrained efficient in the sense above.

Proof. Notice that for the case of single good, an allocation \( x \) is constrained feasible if and only if there is a profile of portfolio \( b \) such that \( x^s_i = e^s_i + r^s b \) for every \( s \) and every \( i \) (see (5.2)). An AR equilibrium is nothing but a competitive equilibrium of a standard \( J + 1 \) commodity general equilibrium model (recall 5.3). By the first fundamental theorem of welfare economics (Proposition 5.1), there is no \( b' \) which makes all the consumers better off, and hence there is no constrained feasible allocation which makes every consumer better off. ■

However, if there are more than one good in each state, one should not expect that an equilibrium is constrained efficient:

Proposition 8.12 If there is more than one good in each state, then a competitive equilibrium is NOT weakly constrained efficient in the sense above, except for non-generic, knife-edge cases.
A formal statement and its proof is mathematically complicated, so I shall not report them here. But the idea behind this result is rather simple, which I explain in the following. Assume for simplicity that the utility functions are additively separable across the states (write $u^s_i$ for the utility in state $s$ for $i$) there is no first period consumption, hence the first period budget constraint is simply $q \cdot b = 0$. Suppose further that after a profile $b$ of portfolios is determined equilibrium prices $\hat{p}^s(b)$ obtain in state $s$. Recall that consumer $i$ is endowed with a bundle of goods $r^s_i b_i + e^s_i$ in state $s$. Let $q^*$ be an equilibrium asset prices and $b^*$ be an equilibrium portfolio of assets.

First, define the indirect utility function for each $i$, in state $s$:

$$v^s_i (p, m) = \max_x u^s_i (x) \text{ subject to } p \cdot x = m.$$ 

Note by the envelope theorem we have Roy’s identity: if $x^*_i \in \mathbb{R}^L$ solves $\max_x u^s_i (x)$ subject to $p \cdot x = m$,

$$\frac{\partial}{\partial m} v^s_i (p, m) x^*_i = - \frac{\partial}{\partial p} v^s_i (p, m) p = Du^s_i (x^*_i)$$ (8.1)

By the definition of equilibrium, if $q^*$ is an equilibrium price vector of asset prices, and if $b^*$ is a profile of an equilibrium portfolio, each consumer $i$ must be choosing portfolio optimally given prices corresponding to $b^*$: that is, $b^*_i$ must solve:

$$\max_{b_i} \sum_s v^s_i (\hat{p}^s (b^*), \hat{p}^s (b^*) \cdot (r^s b_i + e^s_i)) \text{ subject to } q^* \cdot b_i = 0,$$

and hence there is a unique $\beta_i$ such that

$$\sum_s \frac{\partial v^s_i}{\partial m} (\hat{p}^s (b^*) \cdot r^s) = \beta_i q^*$$ (8.2)

Put it differently, this means that asset prices $q^*$ must be equilibrium prices where the future consumption prices are fixed at $\hat{p} (b^*)$. We have already seen this from the single good case in (5.3), where it is automatically the case that the price of goods does not depend on the choices of assets.

The issue is getting clearer now: if constrained efficiency ever fails, it must be due to changes in prices of consumption goods which the consumers do not take into account. Indeed, except for knife edge cases, the prices will change if the portfolios are re-allocated by a social planner, and hence a competitive equilibrium fails to be even constrained efficient. To see this, consider the Pareto problem for a social planner who is interested in improving efficiency:

$$\max \sum_i \xi_i \left( \sum_s v^s_i (\hat{p} (b), \hat{p}^s (b) \cdot (r^s b_i + e_i)) \right) \text{ subject to } \sum_i b_i = 0.$$ 

It can be readily confirmed that if an equilibrium $b^*$ is a constrained efficient allocation, then $b^*$ is a solution to the problem above for some $(\xi_i)_{i=1}^I$. 

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So let’s look at the first order necessary condition of it, and evaluate it at the competitive equilibrium in question. That is, we want to see if the marginal valuations \( \frac{\partial}{\partial b_i} \left( \sum_i \xi_i (\sum_s v_i (\hat{p} (b), \hat{p}^s (b) \cdot (r^s b_i + e_i))) \right) \bigg|_{b=b^*} \) are equated for \( i = 1, ..., I \), if multipliers \( (\xi_i)_{i=1}^I \) are chosen appropriately. More explicitly, using (8.1) we want to see if

\[
\frac{\partial}{\partial b_i} \left( \sum_i \xi_i \left( \sum_s v_i^s (\hat{p} (b), \hat{p}^s (b) \cdot (r^s b_i + e_i^s)) \right) \right) \bigg|_{b=b^*} = \xi_i \left( \sum_s \frac{\partial v_i^s}{\partial m} (\hat{p}^s (b^*) \cdot r^s) \right) + \left( \sum_i \xi_i \left( \sum_s \left( \frac{\partial v_i^s}{\partial p} + \frac{\partial v_i^s}{\partial m} (r^s b_i^* + e_i) \right) \left( \frac{\partial}{\partial b_i} \hat{p}^s (b) \right) \right) \right) = \xi_i \left( \sum_s \frac{\partial v_i^s}{\partial m} (\hat{p}^s (b^*) \cdot r^s) \right) + \left( \sum_k \xi_k \left( \sum_s \left( \frac{\partial v_i^s}{\partial m} (r^s b_k^* + e_k^s - x_k^s) \right) \left( \frac{\partial}{\partial b_i} \hat{p}^s (b^*) \right) \right) \right)
\]

are equated for \( i = 1, ..., I \).

To begin with, suppose that the markets are complete and the equilibrium allocation is efficient. We know that the equilibrium must be equivalent to an AD equilibrium, hence there is \( \lambda_i \) for each \( i \) and prices \( \hat{p} \) such that for each \( i \), \( \lambda_i \hat{p}^s = Du_i^s (x_i^{**}) \) holds for every \( s \). Then from (8.1) we conclude that \( \frac{\partial v_i^s}{\partial m} \) at the equilibrium is independent of \( s \), and so we can choose \( \xi_i \) so that \( \xi_i \frac{\partial v_i^s}{\partial m} = 1 \) for every \( i \) and \( s \). But by market clearing, \( \sum_k (r^s b_k^* + e_k^s - x_k^s) = 0 \) at every \( s \), and hence the last expression in (8.3) is reduced to \( \sum_s (\hat{p}^s (b^*) \cdot r^s) \) which is independent of \( i \). This confirms that an efficient allocation does satisfy the necessary condition we derived above, as it should.

But when the efficiency is not guaranteed so that \( \frac{\partial v_i^s}{\partial m} \) depends on \( s \), equation (8.3) is not readily simplified like this. Using (8.2), the first term in (8.3) which captures the effect from a change in portfolio keeping the prices fixed is reduced to \( \xi_i \beta_i q^s \) for every \( i \) for some positive number \( \beta_i \), so this part can be equated by setting \( \xi_i \beta_i = 1 \). But the remaining part, the effect from change in prices keeping private portfolio fixed, \( \left( \sum_k \xi_k \left( \sum_s \left( \frac{\partial v_i^s}{\partial m} (r^s b_k^* + e_k^s - x_k^s) \right) \left( \frac{\partial}{\partial b_i} \hat{p}^s (b^*) \right) \right) \right) \) are unlikely to be equal \( \xi_i \beta_i = 1 \), since it involves responses to price changes, which affects differently to consumers: for instance, an increase in price of some good will benefit a net seller of that good, but it will deteriorate the welfare of a net buyer of that good. Notice that in the case of single good, the price effect \( \frac{\partial}{\partial b_i} \hat{p}^s (b^*) = 0 \) holds trivially after normalization, which also confirms Proposition 8.11.
Chapter 9

Sunspot Equilibria

Now consider a simple economy with only one asset - a discount bond which pays $1 in the second period. If there is no uncertainty, then the markets are complete. An equilibrium must be Pareto efficient and the number of equilibria is finite generically. Suppose that for some reason, the consumers expected that the second period price (in units of the first period price) will be random. Could this occur in equilibrium and if so will it have any real effect?

To formalize this idea, we imagine the second period price is a random variable with the underlying state space $S$. For ease of exposition let us assume that the underlying states are equally likely. Then we will have a GEI model, with an additional restriction that $e_i^s = e_1^s$ for $s = 1, ..., S$, and the utility function must have an additively separable form $U_i(x_i^0, x_i^1, ..., x_i^S) = \sum_{s=1}^{S} u_i(x_i^0, x_i^s)$.

Clearly there is an equilibrium where consumers expect no randomness of prices, which induces an identical consumption vector. We shall refer to it as a non-sunspot equilibrium. An equilibrium is called a sunspot equilibrium if for some consumer $i$, the equilibrium consumption $x_i = (x_i^0, x_i^1, ..., x_i^S)$ exhibits $x_i^s \neq x_i^{s'}$ for some $s, s' \in \{1, ..., S\}$.

We have a simple welfare characterization for sunspot equilibria.¹

Lemma 9.1 A GEI equilibrium is a sunspot equilibrium if and only if it is Pareto inefficient.

Proof. If a GEI equilibrium is not a sunspot equilibrium, i.e., it is a non-sunspot equilibrium, it must be Pareto efficient by the first fundamental theorem of welfare economics. If a GEI equilibrium is a sunspot equilibrium, then it cannot be Pareto efficient: Let $x$ be a sunspot equilibrium consumption allocation and for every $i$ set $\bar{x}_i^0 = x_i^0$ and $\bar{x}_i^s = \bar{x}_i := \frac{1}{S} \sum_{s'=1}^{S} x_i^{s'}$ for $s = 1, ..., S$. (I.e., $\bar{x}_i$ is the average second period consumption). By the feasibility of the equilibrium consumption vector we have $\sum_i x_i^s = \sum_i e_i^1$ for every $s = 1, ..., S$. Then $\sum_i \bar{x}_i^s = \sum_i \left( \frac{1}{S} \sum_{s'=1}^{S} x_i^{s'} \right) = \frac{1}{S} \left( \sum_{s'=1}^{S} \sum_i x_i^{s'} \right) = \frac{1}{S} \left( \sum_{s'=1}^{S} e_i^1 \right) = \sum_i e_i^1$. Therefore $\bar{x}$ is a feasible consumption allocation and it is as desirable as $x$ for every $i$ by risk aversion. Moreover, a sunspot equilibrium means that $\bar{x}_i \neq x_i$ for at least one $i$, who strictly prefer $\bar{x}_i$ to $x_i$ by strict risk aversion. ■

¹Cass and Shell (1983)
9.1 Existence of sunspot equilibria

Our question is whether or not there exists a sunspot equilibrium. Recall the argument for the indeterminacy result for nominal assets. Starting with a regular equilibrium, one can freely choose the real value $\gamma^s$ of $\$1$ in every state and the real returns of the discount bond is in fact a random variable. So if a non-sunspot equilibrium, which we know exists, is regular, then there will be sunspot equilibria around it and its dimension will be as large as $S - 1$.

But establishing the regularity is tricky. In the perturbation argument we need to freely move endowments but by construction the endowments in the second period need to be the same. Thus the same technique does not work.

An additional issue is that the set of regular economies. In principle, a generic set of regular economies depend on the structure of assets, in particular the number of states. But in the sunspot exercise, the sunspot states are, at least in interpretation, endogenous. So the following result is not trivial.\(^2\)

**Lemma 9.2** There is an open dense subset $\mathcal{E}$ of $(\mathbb{R}^2)^I$ such that for each $e \in \mathcal{E}$, for any number of sunspot states, there are finitely many non-sunspot equilibria, each of which is a regular equilibrium of the sunspot model.

With the generic regularity result above, we can adopt the argument for indeterminacy to establish the existence of sunspot equilibria.\(^3\)

**Proposition 9.3** There is an open dense subset $\mathcal{E}$ of $(\mathbb{R}^2)^I$ such that for each $e \in \mathcal{E}$, for any number $S$ of sunspot states, around any non-sunspot equilibrium, there is a set of sunspot equilibria with dimension $S - 1$.

9.2 Welfare distribution

A sunspot equilibrium is Pareto inefficient, and therefore in each sunspot equilibrium there must be at least one household who is worse off than in the respective efficient equilibrium. Of course, the inefficiency does not imply that all the households are worse off, but it certainly seems plausible and intuitive if this is the case. It may even appear that this will be a prevailing case, since a sunspot equilibrium is “contaminated” by extrinsic, welfare irrelevant randomization by construction, and risk averse households do not appreciate such randomization.

In general, it is true that if the expected real return from an asset is kept constant, increasing the volatility of its returns is welfare worsening to any household. But notice that there is a general equilibrium effect through changing prices, which is overlooked in the observation above. The expected real returns are determined endogenously in equilibrium. In the simple set up we consider where a nominal bond

\(^2\)Cass (1992) establishes the generic regularity, and Kajii (1998) shows that in fact the regularity property is independent of the number of the states.

\(^3\)Basic reference for this result is a special mini-symposium issue of Economic Theory 1992, where the sunspot states are fixed a priori. For the independence of the number of sunspot states is discussed in Kajii (1998).
(inside money) is the only asset, if its average real returns in a sunspot equilibria benefits a particular household, and if the benefit is large enough to offset the loss from the increasing volatility, such a household could gain by sunspots.

Denote by $\gamma^s > 0$ the real return of the nominal discount bond in units of the first period consumption good when the state is $s$; that is, the price of the asset is normalized to be one, and if $b$ units of the asset is held at the end of the first period, $\gamma^s b$ units of consumption good is delivered at the beginning of the second period.

$$-\sum_{s=1}^{S} \frac{\partial}{\partial x_0} u_h \left( e^0_h - b_i, e^1_h + b_i \gamma^s \right) + \sum_{s=1}^{S} \frac{\partial}{\partial x_1} u_h \left( e^0_h - b_i, e^1_h + b_i \gamma^s \right) \gamma^s = 0, \quad (9.1)$$

where $\frac{\partial}{\partial x_0} u_h$ and $\frac{\partial}{\partial x_1} u_h$ are derivatives with respect to the first period consumption and the second period consumption, respectively. The solution is unique if it exists by the strict concavity. The existence depends on the returns and the initial endowments in general, but since our analysis will be done locally around a competitive equilibrium where the optimal choice is well defined, we will simply assume that a solution exists in the relevant domain of the analysis. For a vector of returns $\gamma = (\cdots, \gamma^s, \cdots) \in \mathbb{R}_+^S$, let $Z_h(\gamma)$ be the quantity demanded by household $h$ for the asset. Let $Z(\gamma) := \sum_{i=1}^{l} Z_i(\gamma)$ which is the market excess demand function for the asset.

Recall that we are now normalizing the price of the bond is equal to one so $\gamma$ acts as a vector of prices. An equilibrium is a vector of returns $\gamma = (\cdots, \gamma^s, \cdots) \in \mathbb{R}_+^S$ such that $Z(\gamma) = 0$. A non-sunspot equilibrium corresponds to an equilibrium $\tilde{\gamma} = (\cdots, \tilde{\gamma}, \cdots)$ where the real values are the same across the states.

Assuming the regularity we can solve the equilibrium equation $Z(\gamma) = 0$ implicitly by $\gamma^s, s = 1, \ldots, S-1$. Writing $\gamma_{-S}$ for these elements, we have $Z(\gamma_{-S}, \phi(\gamma_{-S})) = 0$ for any $\gamma_{-S}$ around $\tilde{\gamma}$. We can then write the induced equilibrium excess demand as well as utility level for each $i$:

$$\hat{Z}_i(\gamma_{-S}) := Z_i(\gamma_{-S}, \phi(\gamma_{-S})),
$$

$$\hat{U}_i(\gamma_{-S}) := \sum_{s=1}^{S-1} u_i \left( e^0_i - \hat{Z}_i(\gamma_{-S}), e^1_i + \hat{Z}_i(\gamma_{-S}) \gamma^s \right) + u_i \left( e^0_i - \hat{Z}_i(\gamma_{-S}), e^1_i + \hat{Z}_i(\gamma_{-S}) \phi(\gamma_{-S}) \right).$$

We are then interested in the derivatives of $\hat{U}_i(\gamma_{-S})$, but by the envelope property its first derivatives are zeros. So we need to look into the Hessian of it. Taking advantage of the symmetry of the functions, one can show the following.\footnote{See Kajii (2007).}

**Proposition 9.4** The Hessian matrix $D^2\hat{U}_h(\gamma_{-S})$ is, except for knife edge cases, either positive definite or negative definite. Thus the welfare of consumer $i$ is either maximized or minimized at the non-sunspot equilibrium. Moreover, one of the following two occurs: (1) every borrower’s equilibrium utility $\hat{U}_i(\gamma_{-S})$ is locally maximized at the non-sunspot equilibrium; or (2) every lender’s equilibrium utility is locally maximized.
There are classes of examples where all the households’ level of utility is maximized at the non-sunspot equilibrium. We give one of them as an example below.

**Example 9.5** Let each \( u_h \) be a discounted sum of a quadratic utility function:
\[
u_h(x, y) := (a_h x - x^2) + \delta_h (a_h y - y^2)
\]
where \( a_h > 0 \) and \( \delta_h > 0 \), and assume that a non-sunspot equilibrium exists and the regularity assumptions are satisfied around the non-sunspot equilibrium. By inspection of the first order condition, it can be easily checked that the excess demand function \( Z_h(r) = \alpha_h + \beta_h / (\sum_{s=1}^{S} r^s) \) where \( \alpha_h \) and \( \beta_h \) are constants. Then \( \frac{\partial^2}{\partial r_1 \partial r_2} Z(\bar{r}) = \frac{\partial^2}{\partial (r_1)^2} Z(\bar{r}) \) and so \( \zeta = 0 \) (and \( \xi_h = \frac{\partial^2 u_h}{\partial (x_1)^2} (Z_h(\bar{r}))^2 < 0 \)). For every household, the level of equilibrium utility \( \hat{U}_h(r-S) \) is locally maximized at \( r_S = \bar{r} - S \).

Examples where a household’s utility is locally minimized at the non-sunspot equilibrium can be constructed.

**Example 9.6** Let \( H = 2 \), and set \( u_1(x, y) := \log(x) + \log(y) \) and \( e_1 = (1, 0) \). Then, by direct calculation, we see that for any \( r \gg 0 \), \( Z_1(r) = \frac{1}{2} \). Thus the derivatives of the market excess demand function coincide with those of \( Z_2 \). Given this freedom, it is then possible to find \( Z_2 \) (and underlying utility function \( u_2 \)) such that \( \xi_2 > 0 \). For instance, set \( u_2(x, y) := \log(x) + \log(y) \) and \( e_2 = (0, 1) \), and by direct computations it can be shown that \( \xi_2 > 0 \): this is the leading example of Goenka-Préchac (2006).

Proposition 9.4 does not say which side of the market get benefitted by sunspots. Intuitively, it is likely to be the borrowers in the “usual” equilibrium: sunspots create randomness and by the prudence consumers tends to save more. Which makes the bond price lower, and hence the equilibrium price effect tends to benefit the borrowers.\(^5\)

### 9.3 Production inefficiency

Now suppose that there is production: there is one firm, which will finance its input by the bond. Write \( B \) for the amount of the bond supplied by the firm, and \( z_h \) for the amount held by household \( h \). So the firm raises \( qB \) in units of good in period 0, which is used as input. Consequently, it produces \( f(qB) \) units of good in period 1 and is liable for the outstanding bond, \( B \) dollars. Since the real output is independent of sunspots, the firm’s maximization problem is also unaffected in this setup.

Many insights from the case of pure exchange are then valid in the production economy as well. A sunspot equilibrium exists and indeterminate. There will be gains and losses in sunspot equilibria, especially there will be consumers who are better off at a sunspot equilibrium than in a non-sunspot equilibrium.

We can address the production inefficiency in this context: i.e., compared with a non-sunspot equilibrium benchmark, is the level of production larger or lower?

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\(^5\)Kajii (2007) reports explicit sufficient conditions for this.
Roughly speaking, if consumers are prudent enough, there will be more saving in a sunspot equilibrium, other things being equal. This is so since in a sunspot equilibrium one’s income gets riskier. As we have seen in the leading example, equilibration requires a lower interest rate. Hence the interest rate in the sunspot equilibrium will be lower, which means a higher investment level than the competitive benchmark. In short, sunspots cause over production.

Interestingly enough, if the profit share of the firm can be traded, there is no sunspot equilibrium: there is no equilibrium with a positive production level where consumption is affected by sunspots. The households will trade a “mutual fund” consisting of the bond and the share whose payoff is independent of sunspots. It is also shown that although the returns of the bond and the share might be affected by sunspots, the production level and equilibrium consumption are the same as the textbook case of no sunspot states. That is, if there is a market for the share, in a sharp contrast to the case without it, sunspots do not matter as far as economic welfare is concerned. Of course, if sunspots do not matter, trading profit share is redundant since the price of the share must be determined in such a way that the bond and the share are equivalent as assets in our set up. This is a classical justification of why we do not consider markets for shares in the textbook model of private ownership economies. But in a sunspot equilibrium, this equivalence might break down and so introduction of the profit share market could generate a different kind of sunspot equilibrium\(^6\).

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\(^6\)See Kajii (2009) for the details.
Chapter 10

Appendix

10.1 Separation Theorems

Theorem 10.1 Let $C$ be a non-empty closed convex set in $\mathbb{R}^L$ and $x \notin C$. Then there is $p \in \mathbb{R}^L$ with $p \neq 0$ and a number $\alpha$ such that $p \cdot x < \alpha \leq p \cdot c$ for any $c$.

Proof. Since $C - x$ is a closed convex set not containing 0, it suffices to establish the result for $x = 0$. Since $C$ is closed and $0 \notin C$, there is a non-zero vector $p \in C$ such that $\|p\| = \min \{\|c\| : c \in C\}$. Let $\alpha = \|p\|^2$. Then for any $c \in C$, $tc + (1-t)p = p + t(c-p) \in C$ for any $t \in [0,1]$ since $C$ is convex. By the construction of $p$ and $\alpha$, $\|p + t(c-p)\|^2 = \|p\|^2 + t^2 \|c - p\|^2 + 2tp \cdot (c - p) \geq \alpha$ for any $t \in [0,1]$ and as a function of $t$, the left hand side is minimized at $t = 0$. So its derivative at $t = 0$ must be non-negative. So $2p \cdot (c - p) \geq 0$, i.e., $0 < \alpha \leq p \cdot c$. □

Corollary 10.2 Let $C_1$ and $C_2$ be non-empty closed convex sets in $\mathbb{R}^L$ and $C_1 \cap C_2 = \emptyset$. Assume that at least one of them is compact. Then there is $p \in \mathbb{R}^L$ with $p \neq 0$ and a number $\alpha$ such that $p \cdot c_1 < \alpha \leq p \cdot c_2$ for any $c_1 \in C_1$ and $c_2 \in C_2$.

Proof. Set $C = C_2 - C_1$. $C$ is non-empty and convex, and $0 \notin C$ since $C_1 \cap C_2 = \emptyset$. $C$ is closed since $C_1$ and $C_2$ are closed and at least one of them is compact. Applying the theorem we have $p \in \mathbb{R}^L$ with $p \neq 0$ and a number $\alpha$ such that $0 < \alpha \leq p \cdot (c_2 - c_1)$ for any $c_1 \in C_1$ and $c_2 \in C_2$. Then $p \cdot c_1 < p \cdot c_1 + \alpha \leq p \cdot c_2$. So the result is established by setting $\alpha = \alpha + \sup \{p \cdot c_1 : c_1 \in C_1\}$. □

Corollary 10.3 Let $C$ be a non-empty closed convex set in $\mathbb{R}^L$ such that $C \cap \mathbb{R}^L_{++} = \{0\}$. Then there is $p \in \mathbb{R}^L_{++}$ such that $p \cdot c \leq 0$ for any $c \in C$.

Proof. Let $\Delta := \left\{x \in \mathbb{R}^L_+ : \sum_{i=1}^L x_i = 1\right\}$. $\Delta$ is compact and convex and $C \cap \Delta = \emptyset$. By the corollary above, there is $p \in \mathbb{R}^L$ with $p \neq 0$ and a number $\alpha$ such that $p \cdot c < \alpha \leq p \cdot x$ for any $c \in C$ and $x \in \Delta$. Since $0 \in C$, $\alpha > 0$ must follow. On the other hand, $\min \{p \cdot x : x \in \Delta\} = \min \{p^1, \ldots, p^L\}$, which proves that $p >> 0$. □

For general convex sets, we have a weak separation as follows:

Theorem 10.4 Let $C_1$ and $C_2$ be non-empty convex sets in $\mathbb{R}^L$ and $C_1 \cap C_2 = \emptyset$. Then there is $p \in \mathbb{R}^L$ with $p \neq 0$ such that $p \cdot c_1 \leq p \cdot c_2$ for any $c_1 \in C_1$ and $c_2 \in C_2$. 

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Proof. Set $C = C_2 - C_1$ and then $C$ is a non-empty convex set and $0 \notin C$. It suffices to show that there is $p \in \mathbb{R}^L$ with $p \neq 0$ such that $0 \leq p \cdot c$ for any $c \in C$. Let $F$ be a non-empty finite subset of $C$ and let $\bar{F}$ be its convex hull, i.e., $\bar{F}$ is the set of all convex combinations of points in $F$. By construction $\bar{F}$ is convex and non-empty subset of $C$, and it is compact since $F$ is finite. Also define $F^0 := \{p \in \mathbb{R}^L : \|p\| = 1$ and $p \cdot x \geq 0$ for any $x \in \bar{F}\}$. By construction $F^0$ is a closed set, and by Theorem 10.1, $F^0$ is non empty.

Let $\mathcal{F} := \{F^0 : F$ is a finite subset of $C\}$. Then $\mathcal{F}$ is a collection of closed sub-
sets of a compact set $\{p : \|p\| = 1\}$. Moreover $\mathcal{F}$ exhibits the finite interchapter
property; if $F_1, ..., F_N \in \mathcal{F}$, then $(\bigcup_{i=1}^N F_i)^c \subset \bigcap_{i=1}^N F^0_i$ and since $\bigcup_{i=1}^N F_i$ is finite sub-
set of $C$, $(\bigcup_{i=1}^N F_i)^c$ is non empty, and so is $\bigcap_{i=1}^N F^0_i$. Therefore, there is $p \in \cap \mathcal{F}$. By
construction, $p \neq 0$ and $0 \leq p \cdot c$ for any $c \notin C$. □

10.2 Implicit Function Theorem

10.2.1 preliminary

The (max) norm of a $m \times n$ matrix $A$ is defined by $\|A\| := \max \{\|Ax\| : \|x\| \leq 1\}$. Indeed, $\|A\| \geq 0$, $\|tA\| = t \|A\|$ for any $t > 0$, and $\|A\| = 0$ if and only if $A = 0$. For instance, if $A = (a_1, ..., a_m)$ is an $m$ dimensional vector, $\|A\| = \max \{|a_1|, ..., |a_m|\}$. If $A$ is an $n \times n$ diagonal matrix, $\|A\|$ is the absolute value of the largest diagonal element. Notice by construction $\|Ax\| \leq \|A\| \|x\|$.

Lemma 10.5 [Shrinking function] let $f$ be a function on $S \subset \mathbb{R}^L$ to itself and
suppose that there is a constant $K \in [0, 1)$ such that $\|f(x) - f(y)\| \leq K \|x - y\|$ for any $x, y \in S$. Then there is a unique point $x \in S$ with $x = f(x)$.

Proof. Pick any $x_0 \in S$, and define iteratively that $x_k = f(x_{k-1})$ for $k = 1, 2, ...,$. Then it is readily confirmed that $\{x_k : k = 0, 1, ...\}$ is a Cauchy sequence and hence it has a limit $x^*$. Notice that $\|f(x^*) - f(x_k)\| \leq K \|x^* - x_k\|$ holds for all large enough $n$, hence $\{f(x_k) : k = 0, 1, ...\}$ also converges to $f(x^*)$. Therefore $x^* = f(x^*)$ holds. The uniqueness of a fixed point follows because $f(x) = x$ and $f(y) = y$ means $\|x - y\| = \|f(x) - f(y)\| \leq K \|x - y\|$, which is possible only if $x = y$ since $K < 1$. ■

10.2.2 mean value theorem

For a $C^1$ function $f$ on $\mathbb{R}$, the mean value theorem says that for any $x < y$, $|f(x) - f(y)| \leq (\sup_{z \in [x,y]} f'(z)) |x - y|$. A multidimensional version of this is as follows.

Lemma 10.6 [mean value theorem] Let $U$ be an open set in $\mathbb{R}^n$ and let $x, y \in U$
such that $tx + (1 - t)y \in U$ for any $t \in [0, 1]$. Suppose $f : U \to \mathbb{R}^n$ is $C^1$. Then $\|f(x) - f(y)\| \leq \sup_{t \in [0,1]} \|Df(tx + (1 - t)y)\| \|x - y\|$.

Proof. Consider $t \mapsto f(tx + (1 - t)y)$ and apply the mean value theorem for uni-
variate functions, taking into account the definition of the norm of matrix $\|Df(tx + (1 - t)y)\|$. ■
10.2.3 Implicit function Theorem

Lemma 10.7 (Inverse function theorem) Let $U$ be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ such that $Df (\bar{x})$ is invertible. Then there is a neighborhood $\mathcal{N}$ of $\bar{x}$ such that $f$ is a one to one function on $\mathcal{N}$ and so $f$ has an inverse function. Moreover, the inverse function is $C^r$ and $Df^{-1} (f(x)) = (f(x))^{-1}$.

Proof. Let $A := Df (\bar{x})^{-1}$, then $h(x) := f(A(x - \bar{x}))$ satisfies $h(0) = 0$ and $Dh(x) = I$. It can readily be seen that it is sufficient to establish the result for $h$: that is, we shall show $h$ has a $C^r$ inverse around 0.

Set $g(x) = x - h(x)$. Then $g(0) = 0$ and $Dg(0) = 0$. By the continuity of $x \mapsto \|g(x)\|$, we can choose $\varepsilon > 0$ such that $\|x\| \leq 2\varepsilon$ implies $\|Dg(x)\| \leq \frac{1}{2}$. Let $\mathcal{N} = \{x : \|x\| \leq \varepsilon\}$. Then $x, y \in \mathcal{N}$ means that $\|tx + (1 - t)y\| \leq 2\varepsilon$ for any $t \in [0,1]$, hence $\sup_{t \in [0,1]} \|Dg(tx + (1 - t)y)\| \leq \frac{1}{2}$. So by the mean value theorem (Lemma 10.6), for any $x, y \in \mathcal{N}$, $\|g(x) - g(y)\| \leq \frac{1}{2} \|x - y\|$.

Claim: $h$ is one to one on $\mathcal{N}$. To see this, let $z = h(x)$ and set $\phi(x) = z + g(x)$. Notice that $\phi(\hat{x}) = \hat{x}$ holds if and only if $\hat{x} = z + g(\hat{x}) = z + \hat{x} - h(\hat{x})$, which holds if and only if $z = h(\hat{x})$. So it suffices to show that $\phi$ has a unique fixed point in $\mathcal{N}$. For any $x, y \in \mathcal{N}$, as we have seen above, $\|\phi(x) - \phi(y)\| = \|g(x) - g(y)\| \leq \frac{1}{2} \|x - y\|$, so the shrinking mapping result (Lemma 10.5) shows the uniqueness of a fixed point.

So there exists the inverse function of $h$, denoted by $h^{-1}$. It remains to show that this is $C^r$ in a small neighborhood of 0 ($= h(0)$). Note first that, writing $x = h(x) - g(x)$, $\|x - y\| \leq \|h(x) - h(y)\| + \|g(x) - g(y)\| \leq \|h(x) - h(y)\| + \frac{1}{2} \|x - y\|$, hence $\|x - y\| \leq \frac{1}{2} \|h(x) - h(y)\|$ holds for any $x, y \in \mathcal{N}$.

Let $x$ and $x'$ be in $\mathcal{N}$ and let $y = h(x)$ and $y' = h(x')$. Then $h(x) - h(x') = -Dh(x)(x' - x) - o(x - x')$ and so

$$
\|h^{-1}(y) - h^{-1}(y') - [Dh(x)]^{-1}(y - y')\| = \|x - x' - [Dh(x)]^{-1}(h(x) - h(x'))\| \\
= \|x - x' + (Dh(x)(x' - x) + o(x - x'))\| \\
\leq \|[Dh(x)]^{-1}\| \|o(x - x')\|
$$

Now $\|[Dh(x)]^{-1}\|$ is bounded if $x$ is close enough to 0. Recall that $\|x - x'\| \leq 2 \|h(x) - h(x')\| = 2 \|y - y'\|$, hence $\|o(x - x')\| / \|y - y'\|$ vanishes as $\|y - y'\| \to 0$. Therefore, the inequality above shows that $h^{-1}$ is $C^1$ and $Dh^{-1}(y) = [Dh(x)]^{-1}$.

Notice that the argument above iterates for higher order derivatives, which shows that $h$ is in fact $C^r$. □

Theorem 10.8 (Implicit Function Theorem) Let $U$ be an open set in $\mathbb{R}^m \times \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is $C^r$. Suppose that $D_x f$ is invertible at $(\bar{x}, \bar{y}) \in U \subset \mathbb{R}^m \times \mathbb{R}^n$. Then there is a unique $C^r$ function $\phi$ from a neighborhood $V$ of $\bar{y}$ to $\mathbb{R}^m$ such that $f(\phi(y), y) = f(\bar{x}, \bar{y})$ for all $y \in V$. Moreover, $D\phi(y) = -[D_x f(x,y)]^{-1} D_y f(x,y)$.

Proof. Let $\Phi(x,y) = (f(x,y), y)$. Then $D\Phi = \begin{bmatrix} D_x f & D_y f \\ 0 & I \end{bmatrix}$ is invertible at $(\bar{x}, \bar{y})$. By the inverse function theorem, $\Phi^{-1}$ exists around $(f(\bar{x}, \bar{y}), \bar{y})$ and it is
Write \( \Phi^{-1}(x, y) = (\eta_1(x, y), \eta_2(x, y)) \), and set \( \phi(y) := \eta_1(f(\bar{x}, \bar{y}), y) \). Then by the definition of the inverse, we have \((f(\bar{x}, \bar{y}), y) = \Phi \circ \Phi^{-1}(f(\bar{x}, \bar{y}), y) = \Phi(\eta_1(f(\bar{x}, \bar{y}), y), \eta_2(f(\bar{x}, \bar{y}), y)) = (f(\phi(y), y), \eta_2(f(\bar{x}, \bar{y}), y))\) for any \( y \) around \( \bar{y} \). The first element of this equation is indeed the property we wanted, so \( \phi \) satisfies the requirement.

The uniqueness of such \( \phi \) can be established similarly as in the inverse function theorem.

### 10.2.4 Differentiation on a smooth manifold

A \( C^r \) manifold is an inverse image of a regular \( C^r \) function: Let \( U \) be an open set in \( \mathbb{R}^m \) and \( F : U \to \mathbb{R}^n \) be \( C^r \), where \( m \geq n \), such that rank of \( DF(x) \) is \( n \) whenever \( F(x) = 0 \). Fix such an \( \bar{x} \). By rearranging the coordinates we may as well assume that the first \( n \) column vectors of \( DF(\bar{x}) \) are linearly independent, and write \( x = (x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^{m-n} \). By the implicit function theorem, there is a unique \( C^r \) function \( \phi \) defined locally around \( \bar{x} \) such that \( F(\phi(x_2), x_2) = 0 \) around \( \bar{x}_2 \). That is, at any point in \( F^{-1}(0) \), we can find a local parametrization of points around it where the dimension of the parameters (i.e., \( x_2 \)) is \( m - n \). That is why \( F^{-1}(0) \) is called a \( C^r \) smooth manifold of dimension \( m - n \).

Using the local parametrization as above, we can differentiate a function defined on a smooth manifold. Let \( g \) be a function from \( U \) to \( \mathbb{R}^k \) and we are interested in \( Dg|_{F^{-1}(0)} \). Restricted to \( F^{-1}(0) \), function \( g \) around a point \( \bar{x} \) is given exactly by \( g(\phi(x_2), x_2) \), so the derivative (for this parametrization) is \( \frac{\partial}{\partial x_1}g D\phi + \frac{\partial}{\partial x_2}g = \frac{\partial}{\partial x_1}g \left( -\left( \frac{\partial}{\partial x_1}F \right)^{-1} \frac{\partial}{\partial x_2}F \right) + \frac{\partial}{\partial x_2}g \). We have

**Theorem 10.9** The dimension of the kernel of \( k \times (m - n) \) matrix \( \frac{\partial}{\partial x_1}g \left( -\left( \frac{\partial}{\partial x_1}F \right)^{-1} \frac{\partial}{\partial x_2}F \right) + \frac{\partial}{\partial x_2}g \) is equal to the dimension of kernel of the following \( (k + n) \times (m + n) \) matrix

\[
\begin{bmatrix}
Dg & (DF)^T \\
DF & 0
\end{bmatrix}.
\]

**Proof.** For any \( m-n \) dimensional vector \( \Delta x_2 \), if
\[
\left( \frac{\partial}{\partial x_1}g \left( -\left( \frac{\partial}{\partial x_1}F \right)^{-1} \frac{\partial}{\partial x_2}F \right) + \frac{\partial}{\partial x_2}g \right) \Delta x_2 = 0
\]
then by setting \( \Delta x_1 = -\left( \frac{\partial}{\partial x_1}F \right)^{-1} \frac{\partial}{\partial x_2}F \Delta x_2 \), we have
\[
\begin{bmatrix}
Dg & (DF)^T \\
DF & 0
\end{bmatrix} \begin{bmatrix}
\Delta x_1 \\
\Delta x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
\alpha
\end{bmatrix},
\]
and vice versa. This means that the kernel of the matrix above is a direct sum of the kernel of \( \frac{\partial}{\partial x_1}g \left( -\left( \frac{\partial}{\partial x_1}F \right)^{-1} \frac{\partial}{\partial x_2}F \right) + \frac{\partial}{\partial x_2}g \) and the kernel of \( (DF)^T \). But \( DF \) is a full rank matrix by assumption and so the latter kernel is just zero.
Example 10.10 Differentiate the marginal utility function $DU$ under the constraint $U(x) = \bar{u}$. The corresponding matrix is $\begin{bmatrix} D^2U & DU^T \\ DU & 0 \end{bmatrix}$. We know that the demand function is differentiable if this matrix is invertible. Theorem says that this is equivalent to say $DU$ is a regular mapping on the manifold $U(x) = \bar{u}$.

10.3 Miscellaneous results and proofs

10.3.1 Duality Result

We shall give a proof for Lemma 6.7.

From the standard duality theory, we know that $\bar{S}_i = D_p h_i(p, \bar{u})$ where $h_i$ is the Hicksian demand function (see exercise 6.4) whose differentiability follows from the differentiably strict quasi-concavity. Let $e_i(p, \bar{u}) := p \cdot h_i(p, \bar{u})$ be the minimum expenditure function. Fix a price vector $\bar{p} > 0$. By construction, $e_i(p, \bar{u}) - p \cdot h_i(\bar{p}, \bar{u}) \leq 0$ for any $p$ around $\bar{p}$, and the inequality must be strict unless $p \propto \bar{p}$ in which case the equality must hold. Thus this expression is maximized at $p = \bar{p}$ which shows that $h_i(\bar{p}, \bar{u}) = \frac{\partial}{\partial p} e_i(\bar{p}, \bar{u})$ as well as $p \cdot \frac{\partial}{\partial p} h_i(p, \bar{u}) = 0$ from the first order condition and that $\bar{S}_i = \frac{\partial}{\partial p} h_i(p, \bar{u})$ is negative semi-definite from the second order condition. Moreover, for any non zero vector $a$ with $\bar{p} \cdot \alpha = 0$, $\bar{p} + t \alpha$ is not proportional to $\bar{p}$ for any $t > 0$, hence $0 > e_i(\bar{p} + t \alpha, \bar{u}) - (\bar{p} + t \alpha) \cdot h_i(\bar{p}, \bar{u}) = \left( e_i(\bar{p}, \bar{u}) + t \frac{\partial}{\partial p} e_i(\bar{p}, \bar{u}) \alpha + \frac{t^2}{2} \alpha^T \frac{\partial^2}{\partial p^2} e_i(\bar{p}, \bar{u}) \alpha + o(t^2) \right) - (\bar{p} + t \alpha) \cdot h_i(\bar{p}, \bar{u})$. Taking into account $e_i(\bar{p}, \bar{u}) = \bar{p} \cdot h_i(\bar{p}, \bar{u})$ and $\frac{\partial}{\partial p} e_i(\bar{p}, \bar{u}) = h_i(\bar{p}, \bar{u})^T$, it follows that $0 > \frac{t^2}{2} \alpha^T \frac{\partial^2}{\partial p^2} e_i(\bar{p}, \bar{u}) \alpha + o(t^2)$, hence we conclude that $\alpha^T S_i \alpha \leq 0$ for any non zero vector $a$ with $\bar{p} \cdot \alpha = 0$. In particular, $S_i$ is negative semi definite.

It remains to show that $S_i$’s rank is $L - 1$. This can also be seen by directly finding the inverse of the matrix (6.1): Let $H$ be an orthonormal matrix such that $H^T D^2 U H$ is a diagonal matrix where the diagonal elements are the eigen values of $D^2 U$. The strict differentiable quasi-concavity implies that $L - 1$ of them are negative and one of them is zero. Then it can be readily checked, after rearrangement, $\begin{bmatrix} H & 0 \\ 0 & 1 \end{bmatrix} D^2 U_i \begin{bmatrix} H \\ 0 \end{bmatrix} = \begin{bmatrix} D \end{bmatrix}$, where $D$ is the diagonal matrix of the non-zero eigen values of $D^2 U$. Then $\begin{bmatrix} D \end{bmatrix}^{-1} = \begin{bmatrix} D^{-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. From here it can be seen that $S_i$, which is the top left $L \times L$ matrix of $\begin{bmatrix} D^2 U_i & [DU_i]^T \\ DU_i & 0 \end{bmatrix}$, has rank $L - 1$. 67
10.3.2 The Cass Trick

The analogy of complete market case and GEI case is useful, and there is a convenient technique for establishing the link between aggregate demand functions in the two models, first observed by Cass (often referred to as the Cass trick). Let the consumer 1 can trade as if he is in the Arrow Debreu model: denote by \( \bar{x}_1(p : e_1) \) the Arrow Debreu demand function of consumer 1. For (un-normalized) prices \( p \), write denote \( \hat{p}^s(p) = \frac{1}{p^s} p^s \) and \( q(p) = \frac{1}{p^s} (p^{11}, ..., p^{S1}) R \), that is, the asset prices are calculated so that the price of good one in each state is the state price of that state. Let \( \tilde{Z}_G(p) := (\bar{x}_1(p : e_1) - e_1) + \sum_{i=2}^{I} (x_i(\hat{p}(p), q(p) ; e_i) - e_i) \in \mathbb{R}^L \).

**Lemma 10.11** [Cass Trick] If \( \tilde{Z}_G(p) = 0 \) holds then \( (\hat{p}(p), q(p)) \) is a GEI equilibrium. Conversely if \( (\hat{p}, q) \) is a GEI equilibrium, then \( \tilde{Z}_G(p) = 0 \) where \( p^s = \lambda_1^s \hat{p}_s \), \( \lambda_1^s, s = 1, ..., S \) is consumer 1’s state prices. So there is a one to one correspondence between the GEI equilibria and the equilibria of the modified economy, and the sets of equilibrium consumptions are identical.

**Proof.** When all commodity markets clear, then asset markets clear automatically since \( R \) is full rank, So to confirm the first claim, it suffices to see that consumer 1’s GEI demand and the AD demand coincide, but this follows by construction since consumer 1’s state prices are used to derive the asset prices.

Conversely, starting with a GEI equilibrium, we know that a unique state prices conforming consumer 1’s utility maximization exist, and we know that the demand coincides with the AD demand if prices are weighted by the state prices (Recall Lemma 3.25).

Using this result, the compactness lemma (Lemma 6.8) can be established analogously to the complete market case. Given Lemma 10.11, it suffices to show that \( \{(p, e) : \tilde{Z}_G(p; e) = 0, p^{11} = 1 \text{ and } e \in K\} \subset \mathbb{R}_+^L \times K \) is compact. The set in question is closed by the continuity of \( \tilde{Z} \) and the compactness of \( K \). So we need to show it is bounded.

**Proof.** Suppose that there is a sequence \( (p^n, e^n) \) such that \( \tilde{Z}_G(p^n; e^n) = 0, p^{11,n} = 1 \) and \( e^n \in K \) for every \( n \), and that \( |p^n| \to \infty \). Since \( K \) is compact, we can find \( \bar{e} \in \mathbb{R}_+^L \) such that \( \sum_{i=1}^{I} e^n_i \leq \bar{e} \) for all \( n \). But then \( 0 = \tilde{Z}_G(p^n; e^n) \geq \bar{x}_1(p : e_1) + \sum_{i=2}^{I} x_i(\hat{p}(p^n), q(p^n) ; e_i) - \bar{e} \) must hold for any \( n \) which contradicts the boundary condition for the demand function for consumer 1.

10.4 Fixed Point Theorem

\[
\Delta := \{ x \in \mathbb{R}_+^L : \sum_{l=1}^{L} x_l = 1 \} \\
\Delta^0 := \{ x \in \Delta : x >> 0 \} \\
\partial \Delta := \Delta \setminus \Delta^0 = \{ x \in \Delta : x_l = 0 \text{ for some } l \}
\]
10.4.1 Partition of Unity

**Lemma 10.12** Let $C_1, \ldots, C_K$ be non-empty open sets such that $\Delta = \cup_{k=1}^K C_k$. Then there are non-negative, continuous functions $\alpha_k$ on $\Delta$, $k = 1, \ldots, K$ such that $\sum_{k=1}^K \alpha_k(x) = 1$ for any $x \in \Delta$ and $\alpha_k(x) = 0$ if $x \notin C_k$.

**Proof.** If $C_k = \Delta$ for some $k$, then the result is trivial. So assume that $\Delta \setminus C_k$ is non-empty for every $k$. Let $\delta_k(x) = \inf_{y \in \Delta \setminus C_k} \| x - y \|$ for each $k$. $\delta_k$ is well defined, continuous, non-negative and bounded. Also $\delta_k(x) = 0$ if $x \notin C_k$ by construction, and $\delta_k(x) > 0$ if $x \in C_k$ since $\Delta \setminus C_k$ is closed. Note that for any $x \in \Delta$, $\delta_k(x) > 0$ for at least one $k$ since $\Delta = \cup_{k=1}^K C_k$. Set $\alpha_k(x) = \delta_k(x) / (\sum_{k=1}^K \delta_k(x))$. By construction, each $\alpha_k$ is non-negative, and continuous, and moreover $\sum_{k=1}^K \alpha_k(x) = 1$ for any $x \in \Delta$ and $\alpha_k(x) = 0$ if $x \notin C_k$. ■

10.4.2 Continuous selection

**Lemma 10.13** Let $\phi$ be a set valued function on $\Delta$ such that $\phi(x)$ is a non-empty and convex subset of $\Delta$. Suppose that $\phi^{-1}(y) := \{ x \in \Delta : y \in \phi(x) \}$ is open in $\Delta$ for any $y \in \Delta$. Then there exists a continuous function $f$ from $\Delta$ to itself such that $f(x) \in \phi(x)$ for any $x \in \Delta$.

**Proof.** By assumption $\{ \phi^{-1}(y) : y \in \Delta \}$ is an open cover of $\Delta$, so there are $y_1, \ldots, y_K \in \Delta$ such that $\Delta = \cup_{k=1}^K \phi^{-1}(y_k)$. By partition of unity, there are non-negative, continuous functions $\alpha_k$ on $\Delta$, $k = 1, \ldots, K$ such that $\sum_{k=1}^K \alpha_k(x) = 1$ for any $x \in \Delta$ and $\alpha_k(x) = 0$ if $x \notin \phi^{-1}(y_k)$, i.e., $y_k \notin \phi(x)$. That is, if $\alpha_k(x) > 0$ then $y_k \in \phi(x)$. Define $f(x) := \sum_{k=1}^K \alpha_k(x) y_k$ for each $x \in \Delta$. Then $f$ is a continuous function from $\Delta$ to itself, and $f(x) \in \phi(x)$ for any $x \in \Delta$, since $f(x)$ is a convex combination of points in a convex set $\phi(x)$. ■

10.4.3 Gale-Nikaido Theorem

**Theorem 10.14** Let $Z : \Delta^\circ \rightarrow \mathbb{R}^L$ be a continuous function such that $p \cdot Z(p) = 0$ for any $p \in \Delta^\circ$ (Walras law) and for some positive number $a$, $Z^l(p) \geq -a$ holds for any $l = 1, \ldots, L$ and $p \in \Delta^\circ$ (bounded below). Suppose the following boundary condition holds: for any sequence $p^n$, $n = 1, \ldots$, on $\Delta^\circ$ with $\bar{p} := \lim_n p^n \in \partial \Delta$, there is $l$ with $\bar{p}^l = 0$ such that $Z^l(p^n) \rightarrow \infty$ as $n \rightarrow \infty$. Then there is $p^* \in \Delta^\circ$ with $Z(p^*) = 0$.

**Proof.** First, we establish the following claim: for any sequence $p^n$, $n = 1, \ldots$, in $\Delta^\circ$ with $\bar{p} := \lim_n p^n \in \partial \Delta$, and for any $q \in \Delta$ such that $q^l > 0$ for every $l$ with $\bar{p}^l = 0$, there is $\bar{n}$ such that $q \cdot Z(p^n) > 0$ for every $n > \bar{n}$. To see this, pick $l$ with $\bar{p}^l = 0$ such that $Z^l(p^n) \rightarrow \infty$ as $n \rightarrow \infty$. Then $q \cdot Z(p^n) = q^l Z^l(p^n) + \sum_{i \neq l} q^i Z^i(p^n) > q^l Z^l(p^n) - (L-1)a$. Since $Z^l(p^n) \rightarrow \infty$ and $q^l > 0$, there is $\bar{n}$ such that $q^l Z^l(p^n) > (L-1)a$ for any $n > \bar{n}$.

Define a set valued function $\phi$ on by the following rule:

$$\phi(p) = \begin{cases} 
\{ q \in \Delta : q \cdot Z(p) > 0 \} & \text{if } p \in \Delta^\circ \\
\{ q \in \Delta : q^l > 0 \text{ if } p^l = 0 \} & \text{if } p \in \partial \Delta
\end{cases}$$

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Note that $\phi$ has no fixed point, $p \in \phi(p)$; $p \in \Delta^o$ cannot be a fixed point since $p \cdot Z(p) = 0$, and also $p \in \partial \Delta$ cannot be a fixed point by construction.

We claim that $\phi^{-1}(y)$ is an open set for any $y \in \Delta$. Indeed, let $p \in \phi^{-1}(y)$, i.e., $y \in \phi(p)$. If $p \in \Delta^o$, $y \cdot Z(p) > 0$ holds by the construction of $\phi$ and by the continuity of $Z$, $y \cdot Z(p') > 0$ and hence $p' \in \phi^{-1}(y)$ holds for any $p'$ sufficiently close to $p$. If $p \in \partial \Delta$, then $y' > 0$ for any $l$ with $p' = 0$ by the construction of $\phi$. By the claim shown above, $y \cdot Z(p') > 0$ must hold for any $p' \in \Delta^o$ sufficiently close to $p$. Thus we can find a small enough $\varepsilon > 0$ so that $\|p' - p\| < \varepsilon$ implies that $p'^l > 0$ for every $l$ with $p'^l > 0$ and that $y \cdot Z(p') > 0$ if $p' \in \Delta^o$. Then $p' \in \phi^{-1}(y)$ holds if $\|p' - p\| < \varepsilon$.

The correspondence $\phi$ is clearly convex valued and so by the continuous selection result (Lemma 10.13), if $\phi$ were non-empty valued, then we would find a continuous selection of $\phi$. By Brouwer’s theorem, such a continuous function must have a fixed point which is impossible as we have seen above. Thus there is $p^* \in \Delta$ such that $\phi(p^*) = \emptyset$, which is only possible when $p^* \in \Delta^o$ and $Z(p^*) = 0$. ■
Bibliography


