Equilibrium Prices of the Market Portfolio in the CAPM with Incomplete Financial Markets

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Abstract

In the Capital Asset Pricing Model, we consider how introducing new tradable assets will affect the prices of the existing ones. We prove that introducing new tradable assets into financial markets increases the relative price of the market portfolio with respect to the risk-free bond, or equivalently, decreases the market price of risk, if the elasticity of the marginal rates of substitution of the mean for standard deviation with respect to the latter is greater than one for every consumer; the market price of risk increases if the elasticity is less than one; and the market price of risk is left unchanged if the elasticity is equal to one.

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1 Introduction

In this paper, we consider how introducing new tradable assets into financial markets will affect the prices of the existing ones. Our exercise here is of comparative statics type, whereby we compare two equilibria, one of which is obtained with newly introduced assets and the other without them. For a reason that will soon become clear, we concentrate on the Capital Asset Pricing Model (CAPM), in which, by definition, all consumers believe in the same probability distribution over the states of nature and their utility functions depend only on the mean and standard deviation of random future consumptions. We further assume that consumption takes place only in one period and no consumer has any initial endowments of any tradable asset. Thus all tradable assets under consideration are like futures contracts, that is, for every long position of every tradable asset, there is a short position of that asset, and introducing new tradable assets enhance the risk-hedging opportunities for consumers but does not affect any consumers' endowments. By the "market portfolio," we mean the asset whose payout coincides with the sum of all consumers' initial endowments. By the "risk-free bond," we mean the asset that pays one unit of the good with probability one.

The CAPM admits a very strong characterization of equilibrium asset prices, called the security market line. To be specific, let's index the tradable assets by i and denote the random payout of asset i by m_i and its equilibrium price by p_i. Denote by q_i the return of asset i and by $\overline{q}_i$ its expected return. Denote by d the payout of the market portfolio and define p_d, q_d, and $\overline{q}_d$ to be its price, return, and expected return. The payout of the risk-free bond is denoted by $\mathbf{1}$. Denote its price and return by $p_{\mathbf{1}}$ and $\overline{q}_{\mathbf{1}}$, so that $\overline{q}_{\mathbf{1}} = 1/p_{\mathbf{1}}$. Then, at equilibrium, we must have

$$\overline{q}_i - \overline{q}_{\mathbf{1}} = \beta_i(\overline{q}_d - \overline{q}_{\mathbf{1}}) \tag{1}$$

for every asset i, where

$$\beta_i = \frac{C(q_i, q_d)}{V(q_d)},$$

which is the beta of asset i. We emphasize here that the above equality holds even when the asset markets are incomplete, as long as both the market portfolio and the risk-free bond must are tradable and priced.

It would give rise to unnecessary complications to analyze how the values in (1), such as $\overline{q}_i$, $\overline{q}_d$, and β_i, for existing tradable assets are affected when some new tradable assets are introduced, because they are all defined in terms of returns, and the returns are defined by dividing payouts by prices, the latter of which are endogenously determined at equilibrium. An equivalent condition of (1) in terms of the payouts m_i is that there exist a $t \in \boldsymbol{R}_{++}$ and an $r \in \boldsymbol{R}_{++}$ such that, for every asset i,[1] $p_i = tE(m_i) - rC(m_i, d)$. Since we assume that there is

[1] By plugging $m_i = \mathbf{1}$ and $m_i = d$ and solving the system of two linear equations, we can find that $t = p_{\mathbf{1}}$ and $r = (p_{\mathbf{1}}E(d) - p_d)V(d)^{-1}$. The equivalence of the two representations of the security market line is stated in Duffie (1987).

no consumption in the first period, we can normalize the asset prices p_i so that $t = 1$:

$$p_i = E(m_i) - rC(m_i, d). \tag{2}$$

With this normalization, the risk-free rate is equal to zero and the Sharpe ratio of a portfolio, that is, the expected excess return of the portfolio divided the standard deviation of its return, is equal to $(\bar{q} - 1)/S(q)$, where $\bar{q}$ and $S(q)$ are the mean and standard deviation of the return q of this portfolio. This ratio is maximized when the payout of the portfolio is equal to d (or its scalar multiplication), in which case,

$$\frac{\bar{q} - 1}{S(q)} = \frac{\dfrac{E(d)}{E(d) - rV(d)} - 1}{\dfrac{S(d)}{E(d) - rV(d)}} = \frac{rV(d)}{S(d)} = rS(d).$$

This is called the market price of risk, which is an increasing function of r. The purpose of this paper is to find out under what conditions imposed on utility functions we can unambiguously sign the change in the value of r in (2) when new tradable assets are introduced.

To state our main theorem, index the consumers by h and let, for each h, let U_h be the utility function of consumer h over mean μ and standard deviation σ of random future consumptions satisfying $\partial U_h / \partial \sigma < 0$ and $\partial U_h / \partial \mu > 0$. Define

$$\mathrm{MRS}_h(\sigma, \mu) = -\frac{\partial U_h(\sigma, \mu)/\partial \sigma}{\partial U_h(\sigma, \mu)/\partial \mu}.$$

This marginal rate of substitution measures how much the mean of the consumption should be increased when the standard deviation is increased by one unit, in order for consumer h to enjoy the same utility level. Then the value

$$\frac{\sigma}{\mathrm{MRS}_h(\sigma, \mu)} \frac{\partial\, \mathrm{MRS}_h(\sigma, \mu)}{\partial \sigma} \tag{3}$$

is the elasticity of the marginal rate of substitution with respect to standard deviation σ. This elasticity is thus greater than one if and only if a 1% increase in the standard deviation increases the marginal rate of substitution by more than 1%.

Suppose now that the risk-free bond and the market portfolio (and possibly others) are traded but the markets are incomplete, in the sense that at least one consumer's initial endowments cannot be fully hedged (replicated) by any portfolio of the tradable assets. Then imagine that new tradable assets are introduced into markets and the unhedgeable part of the consumer's initial endowments can be reduced by incorporating some of the new tradable assets in the hedging portfolio. Our main result roughly says that if the elasticity (3) is larger than one at every (σ, μ) and for every consumer h, then, for every equilibrium without the new tradable assets, there is an equilibrium with them at which the value of r in (2) is lower; if the elasticity is smaller than one, then the value of r is higher at some equilibrium with the new tradable assets; and if the elasticity equals one, the value remains the same at some equilibrium with the new tradable assets. If we take for granted the existence and uniqueness of an equilibrium for each set of tradable assets, then we can simply say: if the elasticity (3) is larger than one at every (σ, μ) and for every consumer h, then the value of r in (2) is lower the more complete the asset markets are; if the elasticity (3) is smaller than one at every (σ, μ) and for every consumer h, then the value of r in (2) is higher the more complete the asset markets are; and if the elasticity (3) is equal to one at every (σ, μ) and for every consumer h, then the value of r in (2) is unchanged even when the asset markets become more complete.

To see the applicability of the theorem, consider the following family of utility functions U_h over mean and standard deviation parameterized by $\tau_h > -1$ and $\delta_h > 0$:

$$U_h(\sigma, \mu) = \mu - \frac{\delta_h}{\tau_h + 2}\sigma^{\tau_h + 2}. \tag{4}$$

It is straightforward to check that

$$\frac{\sigma}{\mathrm{MRS}_h(\sigma, \mu)} \frac{\partial\, \mathrm{MRS}_h(\sigma, \mu)}{\partial \sigma} = \tau_h + 1.$$

Hence if $\tau_h > 0$, then introduction of new tradable assets decreases the value of r; if $\tau_h < 0$, then it increases the value of r; and if $\tau_h = 0$, then the value of r remains constant. Recall that the utility function with $\tau_h = 0$, $U_h(\sigma, \mu) = \mu - (\delta_h/2)\sigma^2$, is obtained when an expected utility function exhibits the constant coefficient δ_h of absolute risk aversion and the random future consumptions follow Gaussian distributions. It is well known (shown by Oh (1990, 1996)) that, in this case, the relative prices of existing assets among themselves are not affected by introduction of new tradable assets. Our main theorem thus provides an alternative proof of this well known fact, but it shows more than that: in terms of elasticities of marginal rates of substitution between mean and standard deviation, the well known case is the critical case, above which introduction of new tradable assets increases the price of the market portfolio and below which the former decreases the latter. In particular, the parameter τ_h in the functional form (4) measures the deviation from the case where the market price of risk is unchanged when new tradable assets are introduced.

We should also note that, when it comes to evaluating the effect on the prices of existing assets by introduction of new tradable assets, the marginal rates of substitution $\mathrm{MRS}_h(\sigma, \mu)$, which were shown by Lajeri and Nielsen (2000) to represent degrees of absolute risk aversion over the choice between mean and standard deviation, are not, for themselves, a very helpful piece of information. What is of crucial importance is how much in percentage they will increase as the standard deviation increases. Indeed, in the above example (4), $\mathrm{MRS}_h(\sigma, \mu)$ can be arbitrarily increased or decreased at every (σ, μ) by varying parameter δ_h even when keeping $\tau_h = 0$; but varying δ_h does not affect at all the sign in the change in the value of r when new tradable assets are introduced.

Another example of the unitary elasticity case is

$$U_h(\sigma, \mu) = \mu - \frac{\delta_h}{2}(\sigma^2 + \mu^2)$$

with $\delta_h > 0$. This is obtained when an expected utility function is a quadratic function $u_h(w) = w - (\delta_h/2)w^2$. It is then routine to show that the elasticity of the marginal rates of substitution is always equal to one. Our main theorem again provides an alternative proof of another well-known fact, shown in Oh (1990, 1996), that the relative prices of existing assets among themselves do not depend on the market (in-)completeness when the utility functions are quadratic.

A more general example is the case when U_h is a quadratic function of σ and μ,

$$U_h(\sigma, \mu) = c_h^0\sigma^2 + c_h^1\mu + c_h^2\mu^2,$$

where c_h^0, c_h^1, and c_h^2 are constants. It is again routine to show that the elasticity of the marginal rates of substitution is always equal to one.

Our main result has a nice welfare implication of introduction of new tradable assets. It is often argued that whether it is beneficial to consumers is ambiguous, because the negative pecuniary externality arising from the changes in the prices of existing assets may outweigh the benefit of enhanced risk-hedging opportunities. Suppose now that all consumers have utility functions whose elasticity is larger than one. Suppose also that one of them has an initial risky endowment which perfectly negatively correlated with the market portfolio. Since holding the market portfolio reduces the risk from his initial endowment, his portfolio at equilibrium must consist of a positive amount of the market portfolio and some amount of the risk-free bond. Since the market portfolio is assumed to be traded even before introduction of new tradable assets, they do not enhance his risk-hedging opportunities in any essential way. According to our theorem, the price of the market portfolio goes up as a consequence of introducing new tradable assets. We can thus conclude, with no formal calculation, that this consumer becomes worse off after introduction of new tradable assets.

Our main result requires all consumers to have the elasticities of marginal rates of substitution greater than one, or all having the common elasticity equal to one, or all having elasticities less than one. It does not allow some consumers to have elasticities greater than one and, at the same time, others to have elasticities less than one. It would be nice if we could establish the same sort of predictions on the directions of the change in r when the elasticities greater and less than one coexist. The natural candidate that aggregates different consumers' elasticities is the representative consumer's counterpart. We shall, however, show by means of an example that the representative consumer's elasticity may not provide a correct prediction. Indeed, it is completely possible that his elasticity is greater than or less than one, and yet the value of r may change in either direction depending on the payouts of new tradable assets.

Our main theorem is not a comparative statics exercise assuming the existence of the two equilibria to be compared. Rather, when the existence of an equilibrium is assumed, it establishes the existence of another equilibrium having a particular property in relation to the first one. The proof builds on the beta relation and the mutual fund theorem.

As for the literature, Oh (1990, 1996) obtained the security market line and the mutual fund theorem in the CAPM with incomplete markets, where some consumers' initial endowments cannot be hedged or replicated by the tradable assets. He (and his predecessors referred to in his papers) also proved that introducing new tradable assets does not change the equilibrium prices of the existing ones if the consumers have quadratic utility functions or if they have negative exponential utility functions and consumptions follow Gaussian distributions. Our result not only covers these invariance conditions but predicts the direction of changes in the prices of the market portfolio (the market price of risk) when these conditions are not met.

Dybvig and Ingersoll (1982) gave an example to show that when at least one consumer has a utility function different from quadratic or exponential one and, thus, not leading to mean-variance utility, the beta relation or the asset pricing formula (2) may not hold in the presence of newly introduced tradable assets even when it holds in their absence. While their and our papers both deal with the impact of introducing new tradable assets on the prices of the existing ones, their focus was on the validity of the beta relation, and our focus is on the market price of risk in a model in which the beta relation is satisfied before and after introducing new tradable assets. Detemple and Selden (1991) provided a general equilibrium model similar to but different from the CAPM, in which there are the risk-free bond and a stock (which thus coincides with the market portfolio) initially traded in markets, and introduction of an option on the stock increases the stock price. Our result, however, does not restrict tradable assets to be introduced and, yet, unambiguously predicts the direction of the changes in the prices of the market portfolio. Weil (1992) showed that, under some assumptions on utility functions, the equity premium puzzle of Mehra and Prescott (1985) can be partially solved by the incompleteness of asset markets. His model is different from ours, most importantly, in that only the two polar cases, the complete asset markets and the asset markets consisting only of the risk-free bond and the market portfolio, are compared. Our model can compare two arbitrary asset markets, as long as one can be obtained by adding tradable assets to the other. Note that some economically interesting phenomena, such as some consumers getting worse off as a result of introducing new tradable assets, do not occur when the comparison is restricted to the two polar cases.

Dana (1999) and Hens and Löffler (1996) found that the proof of the existence of an equilibrium in the CAPM with complete markets can be based on the intermediate value theorem, the one-dimensional version of the fixed point theorem. Underlying this approach are the security market line and the mutual fund theorem, because these results allow them to reduce the task of finding an equilibrium to one of solving a single equation by a single unknown.

Elul (1999) showed that if the markets are sufficiently incomplete and there are only a few types of consumers, then it is generically possible to introduce a new tradable asset that leads to a Pareto-improving equilibrium allocation. A key step in his proof is to show that there generically exists a non-redundant asset whose introduction does not change the prices of any existing assets. This property does not hold in our framework. The genericity condition he used refers to a suitably defined set of utility functions and initial endowments, in which utility functions depending only on mean and standard deviation constitute a negligible set. Hara (2011) proved that if there are only finitely many states, S in number, then, regardless of the consumers' preferences or initial endowments, there is a sequence of S assets such that if those S assets are introduced into markets one by one in the order of the sequence, then the asset markets eventually become complete, and every time a new tradable asset is introduced, the prices of the previously introduced ones remain unchanged. This result indicates that the main result of this paper depends crucially on the assumption that the risk-free bond and the market portfolio are always available for trade.

After some earlier versions of this paper were written, Koch-Medina and Wenzelburger (2018), based on Wenzelburger (2010) who assumed that the market portfolio is tradable, published some results that are also included in an earlier version of this paper. Among others, they claimed that their comparative statics result (Proposition 8 of their paper) extended our main result (Theorem 1 of this paper) to the case in which the market portfolio is non-tradable, that is, the market portfolio cannot be hedged or replicated by any portfolio of tradable assets. In Section 4, we give two reasons why, contrary to their claim, their comparative statics result does not really extend our main result, one pertaining to the proof method and the other to the proxy of the market portfolio when it is not tradable.

This paper is organized as follows. Section 2 sets up the model. Section 3 gives the security market line and the mutual fund theorem in incomplete asset markets. Section 4 establishes the main result of this paper. Section 5 shows that it is impossible to use the utility function of the representative consumer to predict the direction of changes in the prices of the market portfolio induced by introduction of new tradable assets. Section 6 concludes, mentioning the possibility of extending the results in this paper to other versions of the CAPM. The proof of the main result is given in the Appendix.

2 The Model

The uncertainty of the economy is described by a probability space Ω. There is only one physical good available in every state and the commodity space X is taken to be the L^2 space over Ω. For simplicity of exposition, throughout this paper, we identify an element of the L^2 space, which is defined to be an equivalent class of random variables that are equal to one another with probability one, with a random variable itself in the equivalent class. The mean $E : X \to \boldsymbol{R}$, variance $V : X \to \boldsymbol{R}$, standard deviation $S : X \to \boldsymbol{R}$, and covariance $C : X \times X \to \boldsymbol{R}$ are defined in the standard way.

Each *consumer*, indexed $h \in \{1, 2, \ldots, H\}$, has a utility function $U_h : \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}$ over the standard deviations and the means of random consumptions. We assume that U_h is twice continuously differentiable and satisfies $\partial U_h(\sigma, \mu)/\partial\sigma \leq 0$ and $\partial U_h(\sigma, \mu)/\partial\mu > 0$ for every $(\sigma, \mu) \in \boldsymbol{R}_+ \times \boldsymbol{R}$; and $\partial U_h(\sigma, \mu)/\partial\sigma < 0$ if $\sigma > 0$ and $\partial U_h(\sigma, \mu)/\partial\sigma = 0$ if $\sigma = 0$. The last assumption is satisfied if U_h is derived from a differentiable expected utility function. Moreover, for every $(\sigma, \mu) \in \boldsymbol{R}_+ \times \boldsymbol{R}$, the Hessian matrix of U_h at (σ, μ),

$$\begin{pmatrix} \dfrac{\partial^2 U_h}{\partial \sigma^2}(\sigma,\mu) & \dfrac{\partial^2 U_h}{\partial \sigma \partial \mu}(\sigma,\mu) \\ \dfrac{\partial^2 U_h}{\partial \sigma \partial \mu}(\sigma,\mu) & \dfrac{\partial^2 U_h}{\partial \mu^2}(\sigma,\mu) \end{pmatrix}, \tag{5}$$

is negative definite on the line orthogonal to the gradient vector $(\partial U_h(\sigma,\mu)/\partial\sigma, \partial U_h(\sigma,\mu)/\partial\mu)$.[2] This condition implies that U_h is strictly quasi-concave.[3] It also has some implications on the domain and differentiability of demand functions, to be seen in the proof of our main theorem (Theorem 1).

Define $W_h : X \to \boldsymbol{R}$ by $W_h(x_h) = U_h(S(x_h), E(x_h))$ for every $x_h \in X$. Then W_h assigns the utility level he obtains from a random consumption $x_h \in X$. The initial endowments of consumer h are denoted by $d_h \in X$. They cannot be directly traded and will be hedged or replicated, possibly only partially, by portfolios of tradable assets, to be defined in the next paragraph. Write $d = \sum_{h=1}^{H} d_h \in X$. This is the aggregate initial endowment. We assume throughout this paper that $V(d) > 0$. In the language of finance, U_h represents consumer h's attitude towards risk and d_h is his initial risk exposure.

For simplicity of exposition, we define a consumer's utility maximization problem and an equilibrium of asset markets directly in terms of market spans and state price functions. A *market span* is a linear subspace M of X, to be understood as the linear subspace spanned by the payouts of the tradable assets; the vectors on M are thus understood as representing the payouts of portfolios. In the literature, the asset markets are often said to be incomplete whenever $M \neq X$, but, in our model, what is more important is whether all the consumers' initial endowments d_h belong to M. Thus, we say that the asset markets are *incomplete* if $d_h \notin M$ for some h, that is, some consumer's initial endowment cannot be hedged or replicated by any portfolio of tradable assets. In other words, even when $M \neq X$, we deem the asset markets complete as long as $d_h \in M$ for every h. LeRoy and Werner (2014, Chapter 19) gave a detailed analysis on non-tradable endowments. In our main result (Theorem 1), we assume that the aggregate initial endowment d belongs to M. We thus call the portfolio with payouts d the *market portfolio*.

A *state price function* is a real-valued linear function $p : M \to R$ on a given market span M. We will be mainly concerned with those in the form $p(m) = E((\mathbf{1} - r(d - E(d)\mathbf{1}))m)$ for every $m \in M$, where $\mathbf{1}$ is the element of X that takes value 1 at every $\omega \in \Omega$, which can be thought of as representing the payout of the *risk-free bond*. As pointed out by Dybvig and Ingersoll (1982), if $r > 0$ is large, the state price density $\mathbf{1} - r(d - E(d)\mathbf{1})$ may take negative values

[2]This is equivalent to

$$\frac{\partial^2 U_h}{\partial \sigma^2}(\sigma,\mu)\left(\frac{\partial U_h}{\partial \mu}(\sigma,\mu)\right)^2 - 2\frac{\partial^2 U_h}{\partial \sigma \partial \mu}(\sigma,\mu)\frac{\partial U_h}{\partial \sigma}(\sigma,\mu)\frac{\partial U_h}{\partial \mu}(\sigma,\mu) + \frac{\partial^2 U_h}{\partial \mu^2}(\sigma,\mu)\left(\frac{\partial U_h}{\partial \sigma}(\sigma,\mu)\right)^2 < 0.$$

This inequality was used in the proof of Lemma 5 of Koch-Medina and Wenzelburger (2018), on which their Propositions 8 and 9 are based.

[3]Unlike Dana (1999), we do not assume strict concavity. Thus, the domain of prices under which there is an optimal portfolio may not be connected, because the consumption set X is not bounded from below.

with positive probabilities. As such, our state price function may admit arbitrage opportunities while observing the law of one price.

The utility maximization problem of consumer h under the market span M is then

$$\begin{aligned} \max_{x_h \in X} \quad & W_h(x_h), \\ \text{s.t.} \quad & x_h - d_h \in M, \\ & p(x_h - d_h) \leq 0. \end{aligned} \tag{6}$$

Note that the linearity of M and p means that there are no transaction costs or short-sales constraints.

We say that a state price function p and a consumption allocation $(x_h^*)_{h \in \{1,2,\ldots,H\}}$ constitute an *equilibrium* under the market span M if, for every h, x_h^* is a solution to the above maximization problem under the market span M, and $\sum_{h=1}^{H} x_h^* = d$.

3 Characterization of an equilibrium in the CAPM

In this section we present, without proof, a proposition on the characterization of an equilibrium when a market span is fixed. This is an intermediate step toward our main theorem, where we compare equilibria under two market spans.

We first introduce some notation. For each subspace M of X, denote by $\operatorname{orth} M$ the orthogonal complement of M, by π_M the orthogonal projection from X onto M, and by P_M the set of all state price functions defined on M. Let $N = \{x \in X \mid E(x) = 0\}$. Then $\pi_N(x) = x - E(x)\mathbf{1}$ for every $x \in X$ and $\pi_N(d)$ is the "de-meaned" market portfolio. Denote by $\mathscr{M}$ the set of all market spans that contain $\mathbf{1}$ and d.

The following characterization theorem is essentially due to Oh (1990, 1996) and can be shown along the lines of Dana (1999). We omit the proof.

Proposition 1 *Let $M \in \mathscr{M}$ and suppose that $p \in P_M$ and $(x_h^*)_{h \in \{1,2,\ldots,H\}} \in X^H$ constitute an equilibrium for M.*

1. *There exist a $t \in \boldsymbol{R}_{++}$ and an $r \in \boldsymbol{R}_{++}$ such that*

$$p(m) = E((t\mathbf{1} - r\pi_N(d))m) \tag{7}$$

   for all $m \in M$.

2. *For every h, there exist an $a_h^* \in \boldsymbol{R}_+$ and a $b_h^* \in \boldsymbol{R}$ such that*

$$x_h^* = a_h^* \pi_N(d) + b_h^* \mathbf{1} + \pi_{\operatorname{orth} M}(d_h).$$

Part 1 of this proposition provides a pricing formula known as the security market line. It implies that the state price density is a strictly positive combination of the risk-free bond $\mathbf{1}$ and the negative of the de-meaned market portfolio, $-\pi_N(d)$. Note that the relative prices of assets on N is invariant to the choice of M. The condition that t be positive is equivalent to saying that the risk-free bond must have a positive price. The condition that r be positive is equivalent to saying that the de-meaned market portfolio $\pi_N(d)$ must have a negative price, which is also equivalent to a strictly positive market price of risk and to a strictly positive slope of the security market line.

Part 2 of Proposition 1 is the mutual fund theorem, with a modification due to incomplete asset markets. It says that every consumer's equilibrium consumption must consist of three terms. The first one is made of the risk-free bond and the second one is made of the market portfolio. Note that everyone holds a non-negative amount of the market portfolio, while some consumers may take a negative amount of (that is, sell short of) the risk-free bond. The third term represents the initial endowment risk that a consumer cannot hedge by trading in the asset markets; this would be zero were the asset markets to be complete.

It is easy to extend Proposition 1 to the case where the market portfolio or the risk-free bond (or either) is not traded.[4]

Proposition 2 *Let M be a linear subspace of X. Suppose that $p \in P_M$ and $(x_h^*)_{h\in\{1,2,\dots,H\}} \in X^H$ constitute an equilibrium for M and that, for every h, there exists an $x_h \in X$ such that $x_h - d_h \in M$ and $W_h(x_h) > W_h(x_h^*)$.*

1. *There exist a $t \in \boldsymbol{R}_{++}$ and an $r \in \boldsymbol{R}_{++}$ such that*

$$p(m) = E((t\mathbf{1} - r\pi_N(d))m)$$

for all $m \in M$.

2. *For every h, there exist an $a_h^* \in \boldsymbol{R}_+$ and a $b_h^* \in \boldsymbol{R}$ such that*

$$x_h^* = a_h^*\pi_{N\cap M}(d) + b_h^*\pi_M(\mathbf{1}) + \pi_{\mathrm{orth}\,M}(d_h).$$

4 Comparative Statics with Variable Market Spans

In this section, we present our main theorem regarding the effect of introducing new tradable assets on the price (and thus the expected return) of the market portfolio. To begin, define $\mathrm{MRS}_h : \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}_+$ by

$$\mathrm{MRS}_h(\sigma,\mu) = -\frac{\partial U_h(\sigma,\mu)/\partial\sigma}{\partial U_h(\sigma,\mu)/\partial\mu}.$$

This marginal rate of substitution measures how much the mean of the consumption should be increased when the standard deviation is increased by one unit, in order to keep the consumer on the same utility level as before. By our assumption on U_h, for every $(\sigma,\mu) \in \boldsymbol{R}_+ \times \boldsymbol{R}$, $\mathrm{MRS}_h(\sigma,\mu) > 0$ if and only if $\sigma > 0$.

[4]Dana (1999, Remark 2.4) gave this result when the risk-free bond is not traded but the market portfolio is.

Definition 1 We say that U_h has *elastic marginal rates of substitution* (EMRS for short) if, for every $(\sigma, \mu) \in \boldsymbol{R}_{++} \times \boldsymbol{R}$,

$$\frac{\sigma}{\mathrm{MRS}_h(\sigma, \mu)} \frac{\partial\, \mathrm{MRS}_h(\sigma, \mu)}{\partial \sigma} > 1.$$

We say that U_h has *inelastic marginal rates of substitution* (IMRS for short) if, for every $(\sigma, \mu) \in \boldsymbol{R}_{++} \times \boldsymbol{R}$,

$$\frac{\sigma}{\mathrm{MRS}_h(\sigma, \mu)} \frac{\partial\, \mathrm{MRS}_h(\sigma, \mu)}{\partial \sigma} < 1.$$

We say that U_h has *unitarily elastic marginal rates of substitution* (UMRS for short) if, for every $(\sigma, \mu) \in \boldsymbol{R}_{++} \times \boldsymbol{R}$,

$$\frac{\sigma}{\mathrm{MRS}_h(\sigma, \mu)} \frac{\partial\, \mathrm{MRS}_h(\sigma, \mu)}{\partial \sigma} = 1.$$

The definition should be clear. The left hand side is the elasticity of the marginal rates of substitution with respect to standard deviations, when the mean is fixed. If the elasticity is larger than one, then a 1% increase in the standard deviation increases the marginal rate of substitution by more than 1%, in which case we say that U_h has elastic marginal rate of substitution. Inelastic and unitarily elastic marginal rates of substitutions are defined analogously.

These definitions are, in fact, stronger than is necessary to establish our main theorem. The slightly weaker conditions, such as the function $\sigma \mapsto \mathrm{MRS}_h(\sigma, \mu)/\sigma$ is strictly increasing, strictly decreasing, or constant for every h and every μ, are sufficient to prove and illustrates it. Yet, we use the above conditions for the simplicity of exposition.

According to Part 1 of Proposition 1, for every $M \in \mathscr{M}$, and for every equilibrium price function $p \in P_M$, there exists an $r \in \boldsymbol{R}_+$ such that $p(m) = E((\mathbf{1} - r\pi_N(d))m)$ for every $m \in M$. For each $r \in \boldsymbol{R}_+$, define $\varphi(r) \in P_X$ by $\varphi(r)(m) = E((\mathbf{1} - r\pi_N(d))m)$. Since $\varphi(r)(d) = E(d) - rV(d)$, the relative price of the market portfolio with respect to the risk-free bond is a decreasing function of r. The market price of risk is nothing but the highest Sharpe ratio, which is attained by the market portfolio and equal to $rS(d)$. Thus, it is an increasing function of r. Define P^*_M as the set of all $r \in \boldsymbol{R}_+$ such that $\varphi(r)$ is an equilibrium state price function for M. Below is the main result of this paper. Its proof is given in the appendix.

Theorem 1 *Let $M \in \mathscr{M}$ and $L \in \mathscr{M}$ and suppose that $M \subseteq L$. Suppose moreover that there are an h and a $z \in L \setminus M$ such that $C(z, d_h) \neq 0$.*

1. *If every U_h has EMRS, then, for every $r^M \in P^*_M$, there exists an $r^L \in P^*_L$ such that $r^M > r^L$.*

2. *If every U_h has EMRS, then, for every $r^L \in P^*_L$, there exists an $r^M \in P^*_M$ such that $r^M > r^L$.*

3. *If every U_h has IMRS, then, for every $r^M \in P^*_M$, there exists an $r^L \in P^*_L$ such that $r^M < r^L$.*

4. If every U_h has IMRS, then, for every $r^L \in P_L^*$, there exists an $r^M \in P_M^*$ such that $r^M < r^L$.

5. If every U_h has UMRS, then $P_M^* = P_L^*$.

This theorem is concerned with the equilibrium prices under two market spans M and L. Imagine, as an example, that the market span M is generated by a set of existing tradable assets and, then, expanded to L as a consequence of introducing new tradable assets into markets. Given a state price function p, the solution x_h^* to the utility maximization problem (6) under M also satisfies the constraints under L. Yet, since a wider range of consumption plans is available under L than under M through trades of new tradable assets, x_h^* may no longer be the solution under L. That is, an expansion of the market span from M to L will change the consumers' demands for the market portfolio, and these changes, in turn, affect the equilibrium asset prices. When the asset prices are given through a state price function $p = \varphi(r)$, a change in equilibrium prices are summarized by a change in the value of r. Recall that a decrease in r means an increase in the price of the market portfolio and a decrease in the market price of risk, and an increase in r means a decrease in the price of the market portfolio and an increase in the market price of risk.

Specifically, part 1 claims that if every U_h has EMRS, then, for every equilibrium before the introduction of the new tradable assets, there exists an equilibrium after the introduction at which the price of the market portfolio is higher. Part 2 claims that, with EMRS, for every equilibrium after the introduction, there exists an equilibrium before the introduction at which the price of the market portfolio is lower. Parts 1 and 2 are equivalent if the equilibria under M and L are unique,[5] in which case, we can simply say that an expansion of the market span increases the price of the market portfolio and decreases the market price of risk. If both sets of equilibria, P_M^* and P_L^*, are compact, then Part 1 is equivalent to $\min P_M^* > \min P_L^*$ and Part 2 is equivalent to $\max P_M^* > \max P_L^*$. In other words, the interval between the highest and lowest equilibrium values of r, $[\min P_M^*, \max P_M^*]$, is a strictly decreasing function of the market span M with respect to the standard order $\geq$ on $\boldsymbol{R}$. The symmetric interpretation can be given to parts 3 and 4. Part 5 says that the set of the equilibrium price functions are not affected by market spans under the assumption of UMRS.

Since $M \subseteq L$, $\operatorname{orth} M \supseteq \operatorname{orth} L$ and, hence, $V(\pi_{\operatorname{orth} M}(d_h)) \geq V(\pi_{\operatorname{orth} L}(d_h))$ for every h. The assumption that there are an h and a $z \in L \backslash M$ such that $C(z, d_h) \neq 0$ means that for some consumer h, a new tradable asset gives an additional hedging opportunity. For this consumer h, we have $V(\pi_{\operatorname{orth} M}(d_h)) > V(\pi_{\operatorname{orth} L}(d_h))$. Thus, the expansion of the market span from M to L does not increases the variance of the unhedgeable part of initial endowments for any consumer, and does indeed decrease the variance for some consumer.

To give an intuition of the proof of the theorem, assume for simplicity that the market market portfolio d has zero mean and unit variance, and consider the case where $L = X$, that

[5] Dana (1999), Hens and Löffler (1996), Hens, Laitenberger, and Löffler (2000) also provided sufficient conditions for the uniqueness of an equilibrium, which are closely related with the condition by Lajeri and Nielsen (2000) for decreasing (or increasing) risk aversion.

is, L represents the complete markets. Suppose that the asset prices are given by $p = \varphi(r)$. When consumer h holds a portfolio of a_h units of the market portfolio and b_h units of the risk-free bond, he receives $a_h d + b_h \mathbf{1}$ from this portfolio. If the asset markets are complete, then he can perfectly hedge his initial endowments and hold positions only in the market portfolio and the risk-free bond. Given the mean-variance utility function U_h and the state price function $p = \varphi(r)$, it is, in fact, optimal for him to do so. Thus the resulting consumption is $a_h d + b_h \mathbf{1}$, with mean b_h and standard deviation a_h, from which he enjoys utility level $U_h(a_h, b_h)$. The first-order condition for optimality is that

$$\mathrm{MRS}_h(a_h, b_h) = r. \tag{8}$$

If, on the other hand, the market span M is incomplete, then he can only hedge his initial endowments up to the point where its remaining part has zero covariance with any consumption plan on the market span M, and it is optimal to do so, given the mean-variance utility function U_h and the state price function $p = \varphi(r)$. The resulting consumption is $a_h d + b_h \mathbf{1} + \pi_{\mathrm{orth}\, M}(d_h)$ with mean b_h and standard deviation $\left(a_h^2 + \theta_h^2\right)^{1/2}$, where $\pi_{\mathrm{orth}\, M}(d_h)$ is the unhedgeable part of the initial endowments (the residual of its orthogonal projection to M) and $\theta_h = S\left(\pi_{\mathrm{orth}\, M}(d_h)\right)$, and the resulting utility level is $U_h\left(\left(a_h^2 + \theta_h^2\right)^{1/2}, b_h\right)$. Under the same state price function $\varphi(r)$ as in the case of complete markets, would it still be optimal to hold the same position (a_h, b_h)? Since, by the chain rule differentiation, the partial derivative of $U_h\left(\left(a_h^2 + \theta_h^2\right)^{1/2}, b_h\right)$ with respect to a_h is equal to

$$\frac{\partial U_h\left(\left(a_h^2 + \theta_h^2\right)^{1/2}, b_h\right)}{\partial \sigma} \frac{a_h}{\left(a_h^2 + \theta_h^2\right)^{1/2}} V(d), \tag{9}$$

and since $p(\pi_N(d))/p(\mathbf{1}) = rV(d)$, the first-order condition for the optimality of (a_h, b_h) is

$$\frac{\mathrm{MRS}_h\left(\left(a_h^2 + \theta_h^2\right)^{1/2}, b_h\right)}{\left(a_h^2 + \theta_h^2\right)^{1/2}} a_h = r. \tag{10}$$

By comparing with (8), we can see that whether (10) holds or not depends on whether the function $\sigma \mapsto \mathrm{MRS}_h(\sigma, \mu)/\sigma$ is constant or not. But this is determined by whether the elasticity is equal to one or not. If the elasticity is equal to one, the first-order condition is still satisfied at the same (a_h, b_h) as in the case of complete markets, and the aggregate demand for the market portfolio remains the same. Thus the equilibrium value of r, the equilibrium price of the market portfolio, and the market price of risk remain the same. This proves part 5 of our main result. If, instead, the elasticity is greater than one, then the function $\sigma \mapsto \mathrm{MRS}_h(\sigma, \mu)/\sigma$ is strictly increasing and the left-hand side of (10) is greater than the right-hand side. This means that in the case of incomplete markets, the demand by every consumer h for the market portfolio is, if well defined, less than a_h, the aggregate demand for the market portfolio is lower, the value of r is higher, the price of the market portfolio is lower, and the market price of risk is higher than in the complete-market case. This is part 1 of the theorem. The other cases can be analogously explained.

Koch-Medina and Wenzelburger (2018) claimed that their Proposition 8 extends Theorem 1 of this paper to the case in which the market portfolio is non-tradable, that is, $d \notin M$. We now give two reasons why, contrary to their claim, the former does not really extend the latter. The first reason is concerned with the proof method, which is valid even when the market portfolio is tradable, and the second reason is concerned with the change in the market price of risk, which is specific to the non-tradable market portfolio.

First, although both Theorem 1 of this paper and Proposition 8 of their paper explore the consequence of changes in variances of the unhedgeable parts of initial endowments, these changes are necessarily discrete in our setting as they are caused by an expansion (or shrinkage) of a market span, while Koch-Medina and Wenzelburger (2018) assumed that they can be continuous, without specifying the cause of these changes. The proof method are accordingly different. The proof of their Lemma 5, on which their Proposition 8 is based, was to apply the implicit function theorem to establish that in the case of EMRS, a continuous increase in the variance of the unhedgeable part of a consumer's initial endowments lowers his demand for the market portfolio, which, in turn, implies that the equilibrium price of the market portfolio is decreased and the market price of risk is increased. Because of the very nature of the implicit function theorem, the result is applicable when the changes in the variance of the unhedgeable parts of initial endowments are small, or when (the changes may be large but) the domain on which the aggregate demand is well defined is connected. However, as explained in Footnote 3, the domain may not be connected, and, since the expansion of the market span may well cause a large change in variances, the variance after the expansion may not belong to the same connected component as the variance before the expansion. The proof method of Proposition 8 of Koch-Medina and Wenzelburger (2018) is, therefore, not applicable to Theorem 1. We prove the theorem by applying the intermediate value theorem, which can deal with large, discrete changes in the variances of the unhedgeable parts of initial endowments, while paying special attention to the possibility that the domain on which the aggregate demand is well defined fails to be connected.

Second, although Proposition 8 of Koch-Medina and Wenzelburger (2018) allows the market portfolio d to be outside the market span M, its price is, then, not defined and we cannot compare the the prices of the market portfolio before and after the change in the market span. We can only compare the prices of the *proxy* of the market portfolio[6] before and after the change in the market span, and the orthogonal projection $\pi_M(d)$ of the market portfolio d on the market span M appears to be the most natural candidate for the proxy, as it is the consumption plan on M that best approximates d. Yet, the change in M typically induces a change in the variance $V(\pi_M(d))$ of the proxy of the market portfolio, which is a cause of the change in the price of the market portfolio that is quite different, in nature, from a change in the risk-hedging opportunities. The only parameter that can be meaningfully compared

[6]Roll (1977) argued that the difficulty to identify the "right" proxy makes it impossible to test the validity of the CAPM.

between two market spans is, therefore, the market price of risk, that is, the highest Sharpe ratio that can be attained by the portfolios of the tradable assets. Recall that Theorem 1 can be restated as a result on the changes in the market price of risk, because the market price of risk is an increasing function of r. As we will see, however, the result no longer holds without the assumption that the market portfolio belongs to the market span. Thus, Proposition 8 of Koch-Medina and Wenzelburger (2018) does not extend Theorem 1.

An example of a single-consumer economy suffices to show that Theorem 1 cannot be extended to the case where the market portfolio is outside the market span. Let the commodity space X be spanned by three elements $\mathbf{1}$, d_0, and d_1, where d_0 and d_1 have zero means, unit variances, and zero covariance. In other words, $\{\mathbf{1}, d_0, d_1\}$ is an orthonormal basis of X. Suppose that the single consumer's utility function U_1 satisfies (4), where $\delta_1 = 1$ but τ_1 is arbitrary. His initial endowment is represented by d_1 (or it can be d_1 plus a positive multiple of $\mathbf{1}$ to guarantee its positive expected return), which is also the market portfolio. Let $\eta \in]0, 1[$ and define $d_\eta = (1 - \eta^2)^{1/2} d_0 + \eta d_1$. Then $E(d_\eta) = 0$, $V(d_\eta) = 1$, and $C(d_1, d_\eta) = \eta$. Let M be the linear subspace of X that is spanned by $\mathbf{1}$ and d_η, and L be the linear subspace that is spanned by $\mathbf{1}$, d_η, and d_1. Then, both M and L contain the risk-free bond $\mathbf{1}$; L contains the market portfolio d_1 but M does not; M is included in L; and $L = X$, that is, the asset markets are complete under the market span L. Since $\pi_M(d_1) = d_\eta$, d_η is the proxy of the market portfolio d_1 under the market span M.

Since the market portfolio d_1 is contained in the market span L, we can define the set P_L^* as in Section 4. On the other hand, the corresponding set for M needs to be carefully defined, as d_1 is not contained in M. Here, we define P_M^* as the set of $r \in \boldsymbol{R}_{++}$ such that the state price function $p(m) = E((\mathbf{1} - r d_\eta) m)$ is an equilibrium state price function for M. We have chosen, in the definition of the state price functional, the proxy d_η of the market portfolio d_1 on M, not the market portfolio d_1 itself, because r should be equal to the market price of risk on M. More precisely, when the state price function is $E((\mathbf{1} - r d_\eta) m)$, the highest Sharpe ratio that can be attained by the consumption plans on M is equal to r, but, when the state price function is $E((\mathbf{1} - r d_1) m)$, the market price of risk, or the highest Sharpe ratio that can be attained by the the consumption plans on M, is equal to ηr. Thus, the coefficient r gives the correct market price of risk in $p(m) = E((\mathbf{1} - r d_\eta) m)$.

We now claim that $P_M^* = \{\eta\}$ and $P_L^* = \{1\}$, that is, writing $r^M = \eta$ and $r^L = 1$, we have $r^M < r^L$ regardless of the value of τ_1. If Theorem 1 were valid for this comparison, then, according to its part 1, we would have $r^M > r^L$ whenever $\tau_1 > 0$. This shows that Theorem 1, without additional assumptions, cannot be extended to the case where the market portfolio is outside the market span.

The proof of $P_M^* = \{\eta\}$ and $P_L^* = \{1\}$ is easy. First, note that the solution to the single consumer's utility maximization problem (6) coincides with d_1, and that $\partial U(S(d_1), E(d_1))/\partial \sigma = -1$ and $\partial U(S(d_1), E(d_1))/\partial \mu = 1$. Second, for every $\alpha \in \boldsymbol{R}$ close to 0, $S(d_1 + \alpha d_\eta) = (1 + 2\eta\alpha + \alpha^2)^{1/2}$ and $E(d_1 + \alpha d_\eta) = 0$. Their derivatives with respect to α evaluated at $\alpha = 0$ are equal to η and 0. Third, since $(-1, 1) \cdot (\eta, 0) = -\eta$, the equilibrium price of d_η is equal to $-\eta$, which implies that $r^M = \eta$. We can similarly show that $r^L = 1$.[7] The proof also shows the logic behind our example: the proxy of the market portfolio becomes a better one as the market span expands; and since the single consumer consumes the market portfolio regardless of the market span, the better the proxy, the lower its price. Since the market price of risk is nothing but the price of the proxy, multiplied by -1, it increases as the market span expands, regardless of the single consumer's utility function.

5 Representative Consumer

In this section we show by means of an example that when some consumers have EMRS and others IMRS, the equilibrium price of the market portfolio may increase or decrease by the introduction of a new tradable asset, depending on its payout structure. This example also shows that, while the elasticity of marginal rate of substitution of mean for standard deviation is a well defined concept even for the representative consumer, it cannot be used to predict the directions of changes in the equilibrium prices of the market portfolio.

To be specific, we consider the parametric family of utility functions that appeared in the introduction:

$$U_h(\sigma, \mu) = \mu - \frac{\delta_h}{\tau_h + 2}\sigma^{\tau_h + 2}. \tag{11}$$

Given the quasi-linearity of these utility functions with respect to mean, we define the representative consumer's utility function as the value function, denoted by W, of the utilitarian social welfare maximization problem, where $x \in X$ is an aggregate consumption:

$$\begin{array}{ll} \max\limits_{(x_h)_h \in X^H} & \sum W_h(x_h) \\ \text{s.t.} & \sum x_h = x. \end{array} \tag{12}$$

We now show that W depends only on mean and standard deviation of x, and quasi-linear with respect to mean. Indeed, let $(x_h^*)_h$ be a solution. Then, for every h, there are a $\sigma_h \in \boldsymbol{R}_+$ and a $\mu_h \in \boldsymbol{R}$ such that $x_h^* = \sigma_h S(x)^{-1}x + \mu_h \mathbf{1}$, that is, x_h^* is a nonnegative multiple of the aggregate endowment, added by a scalar multiple of the payoff of the risk-free bond. In fact, if not, then the orthogonal projections the x_h^* onto the plane spanned by x and $\mathbf{1}$ would attain a higher value of the objective function $\sum W_h(x_h)$. In symbols, let M be the plane spanned by x and $\mathbf{1}$, then $W_h\left(\pi_{\text{orth}\, M}\left(x_h^*\right)\right) \geq W_h\left(x_h^*\right)$ for every h, and it would hold as a strict inequality for some h. Moreover, if $\sigma_h < 0$ for some h, then a transfer of $\varepsilon(S(x))^{-1}x$, where ε is a sufficiently small positive number, from consumer k with $\sigma_k > 0$ to this consumer h would increase both consumers' utility levels. Thus $\sigma_h \geq 0$ for every h. Since $\sum x_h^* = 0$, $\sum \mu_h = 0$, and changing the values of μ_h under this constraint does not change the value of the objective function $\sum W_h(x_h)$ because of the quasi-linearity of U_h with respect to μ. Thus the maximization problem (12) can

[7] The first-order condition (10) is not applicable to the market span M in this example, because (9) does not hold, which is, in turn, because $d \notin M$.

be reduced to the following problem.

$$\begin{aligned} \min_{(\sigma_h)_h \in \boldsymbol{R}_+^H} \quad & \sum \frac{\delta_h}{\tau_h + 2} \sigma_h^{\tau_h + 2}, \\ \text{s.t.} \quad & \sum \sigma_h = S(x). \end{aligned} \tag{13}$$

Denote the value function of this minimization problem by u, and define a utility function U over mean μ and standard deviation σ by $U(\sigma, \mu) = \mu - u(\sigma)$, then $W(x) = U(S(x), E(x))$ and, thus, U represents the representative consumer's utility function W in terms of mean and standard deviation, and is quasi-linear with respect to the mean.

We now show how the representative consumer's marginal rates of substitution of the standard deviation for the mean and their elasticities are related to the individual consumers' counterparts. Denote the representative consumer's marginal rate of substitution, derived from U, by MRS, then it depends on σ but not on μ and satisfies $\mathrm{MRS}(\sigma) = u'(\sigma)$. Thus

$$\frac{\sigma}{\mathrm{MRS}(\sigma)} \frac{\mathrm{d}\,\mathrm{MRS}(\sigma)}{\mathrm{d}\sigma} = \frac{u''(\sigma)\sigma}{u'(\sigma)}, \tag{14}$$

that is, the elasticity of the marginal rate of substitution of U equals the Arrow-Pratt measure of relative risk aversion of u, except that there is no -1 multiplied, because u is convex. The same can be said of for the individual consumers' utility functions U_h. We can thus apply part 2 of Corollary 7 and part 2 of Proposition 15 of Hara, Huang, and Kuzmics (2007) to show that (14) is a decreasing function of σ, starting the largest τ_h and converging to the smallest τ_h. This implies that if the standard deviation $S(x)$ of the aggregate consumption x, which is equal to the market portfolio d at equilibrium, is small, then (14) takes a value close to the largest τ_h at $\sigma = S(x)$, and if $S(x)$ is large, then (14) takes a value close to the smallest τ_h at $\sigma = S(x)$.

Now consider the following example of an economy. There are four consumers. They all have utility functions U_h of the form (11). Consumers $h = 1, 2$ have the same value of τ_h, which is greater than one. Consumers $h = 3, 4$ have the same value of τ_h, which is less than one. Let $\hat{d}$, y^1, y^2 constitute an orthonormal basis of N and define the consumers' initial endowments by

$$\begin{aligned} d_1 &= c(\hat{d} + y^1), \\ d_2 &= c(\hat{d} - y^1), \\ d_3 &= c(\hat{d} + y^2), \\ d_4 &= c(\hat{d} - y^2), \end{aligned}$$

where c is a positive constant. The market portfolio equals $c\hat{d}$, and its standard deviation is equal to c. Assume that the market portfolio and the risk-free bond are initially the only tradable assets.

If c is small, then the standard deviation of the market portfolio is small, and the representative consumer's elasticity is greater than one; if c is large, then the standard deviation of the market portfolio is large, and the representative consumer's elasticity is less than one. If we predicted the direction of changes in the equilibrium prices of the market portfolio based only on the representative consumer's elasticity, then we would conclude that the introduction of new tradable assets will increase the price of the market portfolio when c is small, and it will decrease the price if c is large.

However, if a tradable asset with payout y^1 is introduced, the variances of the unhedgeable parts of the initial endowments of consumers 1 and 2 are decreased but those of consumers 3 are 4 are not. Thus, consumers 1 and 2 trade it, but consumers 3 and 4 do not. Moreover, by $\tau_1 = \tau_2 > 1$, the price of the market portfolio goes up. On the other hand, if a tradable asset with payout y^2 is introduced, consumers 3 and 4 trade it but consumers 1 and 2 do not. Moreover, by $\tau_3 = \tau_4 < 1$, the price of the market portfolio goes down. This is regardless of the values of c. Thus the prediction based on the representative consumer's elasticity is incorrect.

In the above example, the values for the δ_h were not specified. Depending how we specify them, each consumer can be a buyer or seller of the market portfolio. Thus, the incorrect prediction of the equilibrium price of the market portfolio leads to an incorrect prediction of pecuniary externalities and welfare consequences of new tradable assets that we mentioned in the introduction. Jerison (2016) also pointed out the possibility of incorrect prediction of welfare consequences when it is based on the representative consumer's utility function. His point is different from ours in an important respect. While the above example involves the incorrect predictions of both equilibrium prices and welfare consequences, he showed that even when the representative consumer's utility function gives rise to the aggregate demand function of the economy on the entire range of prices and wealth, thereby predicting equilibrium prices correctly, it need not correctly assess the welfare gains or loss associated with a change from the status quo.

6 Conclusion

We have established sufficient conditions in terms of utility functions under which introducing new tradable assets increases or decreases the price of the market portfolio (or the market price of risk) at equilibrium in the CAPM. Those conditions are sufficient to derive the direction of changes in the price of the market portfolio unambiguously regardless of what the new tradable assets under consideration are like. It is also noteworthy that these conditions are on the elasticity of the marginal rates of substitution between mean and standard deviation, not on the marginal rate of substitution itself.

It would be very nice if we could include the non-random first-period consumption in our model and find out conditions under which the relative price between the first-period consumption and the risk-free bond can be affected by introduction of new tradable assets: this is the way Weil (1992) considered his "risk-free rate puzzle." This task, however, seems a rather difficult one when dealing with arbitrary market spans, because we can no longer apply the intermediate value theorem for a comparative statics result, and this is, in turn, because, at least, two relative prices are involved in the analysis, that between the first-period consumption and the risk-free bond and that between the risk-free bond and the market portfolio. To extend our results to the case with the first-period consumption, therefore, it will be necessary to assume that the utility function is separable among the first-period consumptions, the mean of the second-period consumption, and the standard deviation of the second-period consumption; and also that the aggregate demand function has the gross substitute property.

A Proofs

We start with a lemma on the first-order necessary and sufficient condition for the solution to the utility maximization problem (6).

Lemma 1 *Let $M \in \mathscr{M}$, $r \in \boldsymbol{R}_+$, and $x_h \in X$. Then x_h solves the utility maximization problem (6) when $p = \varphi(r)$ if and only if there is an $(a_h, b_h) \in \boldsymbol{R}_+ \times \boldsymbol{R}$ such that*

$$x_h = a_h \pi_N(d) + b_n \mathbf{1} + \pi_{\mathrm{orth}\, M}(d_h) \tag{15}$$

$$b_h - rV(d)a_h = E(d_h) - rC(d, d_h), \tag{16}$$

$$r = \frac{\mathrm{MRS}_h\left(\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\, M}\left(d_h\right)\right)\right)^{1/2}, b_h\right)}{\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\, M}\left(d_h\right)\right)\right)^{1/2}} a_h. \tag{17}$$

This lemma implies that x_h solves the utility maximization problem (6) when $p = \varphi(0)$, that is, $r = 0$, if and only if $x_h = E(d_h)\mathbf{1} + \pi_{\mathrm{orth}\, M}(d_h)$, that is, $a_h = 0$ and $b_h = E(d_h)$.

Proof of Lemma 1 Suppose first that x_h solves the utility maximization problem (6) when $p = \varphi(r)$. The first equality (15) can be derived just like part 2 of Proposition 1. Then $a_h \geq 0$. By plugging it into the budget constraint with equality, $\varphi(r)(x_h - d_h) = 0$, we obtain (16). To prove (17), note first that if (15) is met, then $S(x_h) = \left(a_h^2 V(d) + V\left(\pi_{\mathrm{orth}\, M}(d_h)\right)\right)^{1/2}$ and $E(x_h) = b_h$. Thus,

$$W_h\left(a_h \pi_N(d) + b_n \mathbf{1} + \pi_{\mathrm{orth}\, M}(d_h)\right) = U_h\left(\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\, M}(d_h)\right)\right)^{1/2}, b_h\right).$$

Hence,

$$\frac{\mathrm{d}W_h}{\mathrm{d}a_h}((a_h \pi_N(d) + b_n \mathbf{1} + \pi_{\mathrm{orth}\, M}(d_h)) = \frac{\dfrac{\mathrm{d}U_h}{\mathrm{d}\sigma}\left(\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\, M}(d_h)\right)\right)^{1/2}, b_h\right)}{\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\, M}\left(d_h\right)\right)\right)^{1/2}} V(d)a_h, \tag{18}$$

$$\frac{\mathrm{d}W_h}{\mathrm{d}b_h}((a_h \pi_N(d) + b_n \mathbf{1} + \pi_{\mathrm{orth}\, M}(d_h)) = \frac{\mathrm{d}U_h}{\mathrm{d}\mu}\left(\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\, M}(d_h)\right)\right)^{1/2}, b_h\right), \tag{19}$$

The first-order necessary and sufficient condition for $x_h = a_h \pi_N(d) + b_n \mathbf{1} + \pi_{\mathrm{orth}\, M}(d_h)$ to solve the utility maximization problem is that

$$rV(d) \geq -\frac{\dfrac{\mathrm{d}W_h}{\mathrm{d}a_h}((a_h\pi_N(d) + b_n\mathbf{1} + \pi_{\mathrm{orth}\,M}(d_h))}{\dfrac{\mathrm{d}W_h}{\mathrm{d}b_h}((a_h\pi_N(d) + b_n\mathbf{1} + \pi_{\mathrm{orth}\,M}(d_h))},$$

which holds as a strict inequality if $a_h > 0$. But since $\partial U_h(0,\mu)/\partial\sigma = 0$ for every $\mu \in \boldsymbol{R}$, this holds as an equality even when $a_h = 0$, in which case $r = 0$. By plugging (18) and (19) into the left-hand side of the above equality, we obtain (17).

Suppose conversely that there is an $(a_h, b_h) \in \boldsymbol{R}_+ \times \boldsymbol{R}$ for which (15), (16), and (17) hold. Since U_h is quasi-concave, so is W_h. Thus the three equalities imply that it is impossible to increase $W_h\left(a_h\pi_N(d) + b_n\mathbf{1} + \pi_{\mathrm{orth}\,M}(d_h)\right)$ by choosing other values of (a_h, b_h) subject to (16). Since (16) is equivalent to $\varphi(r)(x_h - d_h) = 0$, x_h is the solution to the utility maximization problem (6) with the additional constraint that $x_h = a_h\pi_N(d) + b_n\mathbf{1} + \pi_{\mathrm{orth}\,M}(d_h)$ for some (a_h, b_h). But since the solution to (6) must be in this form for some (a_h, b_h), x_h is, in fact, the solution to (6). ///

For each h and $M \in \mathscr{M}$, let T_h^M be the set of those $r \in \boldsymbol{R}_+$ for which the maximization problem (6) has a solution under $p = \varphi(r)$ and, for each $r \in T_h^M$, denote the (a_h, b_h) as in (15) by $(a_h^M(r), b_h^M(r)) \in \boldsymbol{R}_+ \times \boldsymbol{R}$. We have thus defined a function $a_h^M : T_h^M \to \boldsymbol{R}_+$. By Lemma 1, $0 \in T_h^M$, and, for every $r \in T_h^M$, $a_h^M(r) = 0$ if and only if $r = 0$.

Lemma 2 *Let $M \in \mathscr{M}$.*

1. *T_h^M is an open subset of $\boldsymbol{R}_+$ and a_h^M is continuously differentiable.*

2. *Let $r \in \boldsymbol{R}_+$ and $(a_h, b_h) \in \boldsymbol{R}_+ \times \boldsymbol{R}$, and suppose that $b_h - rV(d)a_h = E(d_h) - rC(d, d_h)$. Then $r \in T_h^M$ and $a_h^M(r) \leq a_h$ if and only if*

$$r \leq \frac{\mathrm{MRS}_h\left(\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}, b_h\right)}{\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}} a_h. \tag{20}$$

3. *For every $r \in \boldsymbol{R}_+ \setminus T_h^M$ and for every sequence (r^n) in T_h^M, if $r^n \to r$, then $a_h^M(r^n) \to \infty$.*

Part 1 of this lemma is clear. Part 3 is the so-called boundary behavior of the demand function for the market portfolio. It states that as the price of the market portfolio, at which consumer h's demand is well defined, approaches a level at which consumer h's demand is not well defined, his demands for the market portfolio become large without bounds. It is similar to part (iii) of Lemma 2 of Koch-Medina and Wenzelburger (2018) but different from it in that we do not have to compare the level r with the asymptotic slope of an indifference curve of consumer h. Part 2 gives an equivalent condition for $r \in T_h^M$ and, in addition, an upper bound on the demand for the market portfolio. It implies that if $r \in T_h^M$ and (20) fails to hold, then $a_h^M(r) > a_h$. If (20) holds as an equality, then $a_h^M(r) = a_h$ by Lemma 1. Thus, part 2 implies that if $r \in T_h^M$ and

$$r \geq \frac{\mathrm{MRS}_h\left(\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}, b_h\right)}{\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}} a_h,$$

then $a_h^M(r) \geq a_h$. This fact will also be used in the proof of Theorem 1.

Proof of Lemma 2 Define $U_h^M : \boldsymbol{R}_+ \times \boldsymbol{R}$ by letting

$$U_h^M(a_h, b_h) = U_h\left(\left(V(d)a_h^2 + \pi_{\mathrm{orth}\,M}(d_h)\right)^2, b_h\right)$$

for every (a_h, b_h). Then U_h^M satisfies the condition on the Hessian matrix (5) when U_h is replaced by U_h^M. Thus, by Proposition 2.7.2 of Mas-Colell (1985), the demand function, for (a_h, b_h) derived from U_h^M is defined on an open subset of the price space $\boldsymbol{R}_+ \times \boldsymbol{R}_{++}$ and, on the subset, the demand function is continuously differentiable. Since $U_h^M(a_h, b_h) = W_h\left(a_h \pi_N(d) + b_n \mathbf{1} + \pi_{\mathrm{orth}\,M}(d_h)\right)$ for every (a_h, b_h), this establishes part 1.

Part 2 is true when $r = 0$, because, then, $E(d_h)\mathbf{1} + \pi_{\mathrm{orth}\,M}(d_h)$ solves the utility maximization problem (6). Suppose that $r > 0$. Suppose also that (20) holds and consider the function

$$a \mapsto \frac{\mathrm{MRS}_h\left(\left(V(d)a^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}, E(d_h) - r(C(d, d_h) - V(d)a)\right)}{\left(V(d)a^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}} a - r.$$

This function takes a negative value when $a = 0$ and a nonnegative value when $a = a_h$. Thus, by the intermediate value theorem, there is an $a \in]0, a_h]$ at which this function takes zero. Then, by Lemma 1, the utility maximization problem is solved by $a\pi_N(d) + (E(d_h) - r(C(d, d_h) - V(d)a))\mathbf{1} + \pi_{\mathrm{orth}\,M}(d_h)$. Thus $r \in T_h^M$ and $a = a_h^M(r) \leq a_h$. Suppose conversely that $r \in T_h^M$ and $a_h^M(r) \leq a_h$. If $a_h^M(r) = a_h$, then, by Lemma 1, (20) holds with an equality. Suppose that $a_h^M(r) < a_h$. Then $b_h^M(r) < b_h$ as well. Since $U_h^M\left(a_h^M(r), b_h^M(r)\right) \geq U_h^M\left(a_h, b_h\right)$ and U_h^M is quasi-concave,

$$\frac{\partial U_h^M}{\partial a_h}(a_h, b_h)\left(a_h^M(r) - a_h\right) + \frac{\partial U_h^M}{\partial b_h}(a_h, b_h)\left(b_h^M(r) - b_h\right) \geq 0,$$

that is,

$$-\frac{\dfrac{\partial U_h^M}{\partial a_h}(a_h, b_h)}{\dfrac{\partial U_h^M}{\partial b_h}(a_h, b_h)} \geq \frac{b_h^M(r) - b_h}{a_h^M(r) - a_h}.$$

By (18) and (19), the left-hand side is equal to

$$\frac{\mathrm{MRS}_h\left(\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}, b_h\right)}{\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}} V(d)a_h.$$

Since $b_h - rV(d)a_h = E(d_h) - rC(d, d_h) = b_h^M(r) - rV(d)a_h^M(r)$, the right-hand side is equal to $rV(d)$. Plugging these results, we obtain (20).

We prove part 3 by contradiction. Suppose that the conclusion were false. Then there would be an $r \in \boldsymbol{R}_+ \setminus T_h^M$, a sequence (r^n) in T_h^M, and an $a_h \in \boldsymbol{R}_+$ such that $r^n \to r$ and $a_h^M(r^n) \to a_h$. Then $b_h^M(r^n) \to E(d_h) - r(C(d, d_h) - V(d)a_h)$ and $(a_h, E(d_h) - r(C(d, d_h) - V(d)a_h))$ satisfies (15), (16), and (17). By Lemma 1, this means that $a_h\pi_N(d) + b_n\mathbf{1} + \pi_{\text{orth}\,M}(d_h)$ solves the utility maximization problem. This is a contradiction to the assumption that $r \in \boldsymbol{R}_+ \setminus T_h^M$. ///

Define $T^M = \bigcap_{h=1}^H T_h^M$. Then T^M consists of those $\in \boldsymbol{R}_+$ for which the aggregate demand $\sum_{h=1}^H a_h^M(r)$ is well defined. Define the aggregate demand function $a^M : T^M \to \boldsymbol{R}_+$ by $a^M(r) = \sum_{h=1}^H a_h^M(r)$. Then $0 \in T^M$, and, for every $r \in T^M$, $a^M(r) = 0$ if and only if $r = 0$. For each $r \in T^M$, let

$$x_h^M = a_h^M(r)\pi_N(d) + \left(E(d_h) - r(C(d, d_h) - V(d)a_h^M(r)\right)\mathbf{1} + \pi_{\text{orth}\,M}(d_h),$$

then, by $d \in \mathscr{M}$,

$$\begin{aligned}\sum_h x_h^M &= a^M(r)\pi_N(d) + E(d) - r\left(C(d,d) - V(d)a^M(r)\right)\mathbf{1} + \pi_{\text{orth}\,M}(d) \\ &= a^M(r)\pi_N(d) + \left(E(d) - rV(d)(1 - a^M(r))\right)\mathbf{1}.\end{aligned}$$

Thus, $\sum_h x_h^M = d$ if and only if $a^M(r) = 1$. Hence, $r \in P_M^*$ if and only if $a^M(r) = 1$.

Proof of Theorem 1

1. We consider two cases according to whether $[0, r^M] \subseteq T^L$ or not.

   First, consider the case where $[0, r^M] \not\subseteq T^L$.[8] Since $0 \in T^M$, part 1 of Lemma 2 implies that there is a (unique) $r \in \,]0, r^M]$ such that $[0, r[\, \subseteq T^L$ and $r \notin T^L$. Then there is an h such that $r \notin T_h^L$. By part 3 of Lemma 2, for every sequence $(r^n)_n$ in T_h^L, if $r^n \to r$ as $n \to \infty$, then $a_h^L(r^n) \to \infty$ as $n \to \infty$. Thus, $a^L(r^n) \to \infty$ as $n \to \infty$. Since $a^L(0) = 0$, the intermediate value theorem implies that there is an $r^L \in \,]0, r[$ such that $a^L(r^L) = 1$. Thus $r^L < r^M$ and $r^L \in P_L^*$.

   Next, consider the case where $[0, r^M] \subseteq T^L$. Since $M \subset L$ and $V\left(\pi_{\text{orth}\,M}(d_h)\right) \geq V\left(\pi_{\text{orth}\,L}(d_h)\right)$ for every h, and it holds as a strict inequality for some h, because there are an h and a $z \in L \setminus M$ such that $C(z, d_h) \neq 0$. Since U_h has EMRS for every h, by (17),

$$r^M \geq \frac{\text{MRS}_h\left(\left(V(d)\left(a_h^M(r^M)\right)^2 + V\left(\pi_{\text{orth}\,L}(d_h)\right)\right)^{1/2}, b_h^M(r^M)\right)}{\left(V(d)\left(a_h^M(r^M)\right)^2 + V\left(\pi_{\text{orth}\,L}(d_h)\right)\right)^{1/2}} a_h^M(r^M), \tag{21}$$

[8] This is the case that was not covered by the proof of Lemma 5 of Koch-Medina and Wenzelburger (2018), on which their Proposition 8 is based. An analogous case appears in the proofs of Parts 2, 3, and 4 as well.

and it holds as a strict inequality for some h. Since $r^M \in T^L$, by (20), $a_h^L(r^M) \geq a_h^M(r^M)$ for every h, and it holds as a strict inequality for some h. Thus $a^L(r^M) > a^M(r^M) = 1$. Hence, by the intermediate value theorem, there is an $r^L \in]0, r^M[$ such that $a^L(r^L) = 1$. Thus $r^L < r^M$ and $r^L \in P_L^*$.

2. Just like (21), we can show that

$$r^L \leq \frac{\text{MRS}_h \left(\left(V(d) \left(a_h^L(r^L) \right)^2 + V\left(\pi_{\text{orth}\, M}\left(d_h\right)\right) \right)^{1/2}, b_h^L(r^L) \right)}{\left(V(d) \left(a_h^L(r^L) \right)^2 + V\left(\pi_{\text{orth}\, M}\left(d_h\right)\right) \right)^{1/2}} a_h^L(r^L)$$

for every h, with a strict inequality for some h. By part 2 of Lemma 2, $r^L \in T_h^M$ and $a_h^M(r^L) \leq a_h^L(r^L)$ for every h, with a strict inequality for some h. Thus $r^L \in T^M$ and $a^M(r^L) < a^L(r^L) = 1$.

We consider two cases according to whether $[r^L, \infty[\subseteq T^M$ or not. If $[r^L, \infty[\not\subseteq T^M$, then, by part 3 of Lemma 2, there is a (unique) $r \in]r^L, \infty[$ such that $[r^L, r[\subseteq T^M$ and $r \notin T^M$. Then, as we saw in the proof of part 1 of this theorem, for every sequence $(r^n)_n$ in T^M, if $r^n < r$ for every n and $r^n \to r$, then $a^L(r^n) \to \infty$. Since $a^M(r^L) < 1$, the intermediate value theorem implies that there is an $r^M \in]r^L, r[$ such that $a^M(r^M) = 1$. Thus $r^M > r^L$ and $r^L \in P_L^*$. If $[r^L, \infty[\subseteq T^M$, let $r \in \boldsymbol{R}_+$ satisfy $r > r^L$ and, for every h,

$$r \geq \frac{\text{MRS}_h \left(\left(V(d) a_h^2 + V\left(\pi_{\text{orth}\, M}\left(d_h\right)\right) \right)^{1/2}, b_h \right)}{\left(V(d) a_h^2 + V\left(\pi_{\text{orth}\, M}\left(d_h\right)\right) \right)^{1/2}} a_h$$

where $(a_h, b_h) = (C(d, d_h)/V(d), E(d))$. Since $r \in T^M$, part 2 of Lemma 2 implies that $a_h^M(r) \geq C(d, d_h)/V(d)$ for every h. Thus $a^M(r) \geq 1$.[9] Thus, by the intermediate value theorem, there is an $r^M \in]r^L, r]$ such that $a^M(r^M) = 1$. Thus $r^M > r^L$ and $r^M \in P_M^*$.

3. The proof of this part is analogous to that of part 2. We can show that

$$r^M \leq \frac{\text{MRS}_h \left(\left(V(d) \left(a_h^M(r^M) \right)^2 + V\left(\pi_{\text{orth}\, L}\left(d_h\right)\right) \right)^{1/2}, b_h^M(r^M) \right)}{\left(V(d) \left(a_h^M(r^M) \right)^2 + V\left(\pi_{\text{orth}\, L}\left(d_h\right)\right) \right)^{1/2}} a_h^M(r^M)$$

for every h, with a strict inequality for some h. Thus $r^M \in T^L$ and $a^L(a^M) < 1$.

We consider two cases according to whether $[r^M, \infty[\subseteq T^L$ or not. If $[r^M, \infty[\not\subseteq T^L$, then there is a (unique) $r \in]r^M, \infty[$ such that $[r^M, r[\subseteq T^L$ and $r \notin T^L$. Then, for every sequence $(r^n)_n$ in T^L, if $r^n \to r$, then $a^L(r^n) \to \infty$. Hence there is an $r^L \in]r^M, r[$ such that $a^L(r^L) = 1$. Thus $r^L > r^M$ and $r^L \in P_L^*$. If $[r^M, \infty[\subseteq T^L$, let $r \in \boldsymbol{R}_+$ satisfy $r > r^M$

[9] The proof of Theorem 3 of Koch-Medina and Wenzelburger (2018) contained this argument. An analogous argument appears in Part 3 as well.

and, for every h,

$$r \geq \frac{\mathrm{MRS}_h\left(\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\,L}\left(d_h\right)\right)\right)^{1/2}, b_h\right)}{\left(V(d)a_h^2 + V\left(\pi_{\mathrm{orth}\,L}\left(d_h\right)\right)\right)^{1/2}} a_h$$

where $(a_h, b_h) = (C(d, d_h)/V(d), E(d))$. Then $a^L(r) \geq 1$. Thus, there is an $r^L \in]r^M, r]$ such that $a^L(r^L) = 1$. Thus $r^L > r^M$ and $r^L \in P_L^*$.

4. The proof of this part is analogous to that of part 1. We consider two cases according to whether $[0, r^L] \subseteq T^M$ or not.

   First, consider the case where $[0, r^L] \not\subseteq T^M$. Then there is a (unique) $r \in]0, r^L]$ such that $[0, r[\subseteq T^M$ and $r \notin T^M$. Then, for every sequence $(r^n)_n$ in T^M, if $r^n \to r$ as $n \to \infty$, then $a^M(r^n) \to \infty$ as $n \to \infty$. Hence, there is an $r^M \in]0, r[$ such that $a^M(r^M) = 1$. Thus $r^M < r^L$ and $r^M \in P_M^*$.

   Next, consider the case where $[0, r^L] \subseteq T^M$. We can show that

$$r^L \geq \frac{\mathrm{MRS}_h\left(\left(V(d)\left(a_h^L(r^L)\right)^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}, b_h^L(r^L)\right)}{\left(V(d)\left(a_h^L(r^L)\right)^2 + V\left(\pi_{\mathrm{orth}\,M}\left(d_h\right)\right)\right)^{1/2}} a_h^L(r^L),$$

   and it holds as a strict inequality for some h. Thus $a^M(r^L) > 1$. Hence, by the intermediate value theorem, there is an $r^M \in]0, r^L[$ such that $a^M(r^M) = 1$. Thus $r^M < r^L$ and $r^M \in P_M^*$.

5. Since every U_h has UMRS, for every $M \in \mathscr{M}$, every $L \in \mathscr{M}$, $r \in \boldsymbol{R}_+$, and $(a_h, b_h) \in \boldsymbol{R}_+ \times \boldsymbol{R}$, (17) holds if and only if it holds when M is replaced by L. Thus $a^M(r) = a^L(r)$ and, hence, $P_M^* = P_L^*$.

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