Effectively Complete Asset Markets with Multiple Goods and over Multiple Periods

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Abstract

Following LeRoy and Werner (2001), we propose a definition of effectively complete asset markets in a model with multiple goods and multiple periods, and establish the first welfare theorem in such markets. As applications of the theorem, we derive the Pareto-efficiency of equilibrium allocation in economies with no aggregate risk and the mutual fund theorem. We also extend the sunspot irrelevance theorem of Mas-Colell (1992) to the model of multiple periods and the no-retrade theorem of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003) to the case where the asset prices need not be time-invariant Markov processes.

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1 Introduction

Asset markets are said to be *complete* if any pattern of transfers of contingent commodities across states and over time can be financed by trading assets. In the case of two consumption periods, with no uncertainty on the first period and S possible states of the world on the second, asset markets are complete if and only if there are S non-redundant assets. In complete asset markets, the equilibrium allocations are Pareto-efficient.

If we impose some restrictions utility functions and initial endowments, then we may narrow down the class of Pareto-efficient allocations, and hence the class of patterns of transfers of contingent commodities attaining Pareto-efficient allocations. It might even be true,

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in the case of two consumption periods, that with fewer than S non-redundant assets, the equilibrium allocations are Pareto-efficient, as in the case of complete asset markets.

LeRoy and Werner (2001, Section 16.3) made this observation precise by giving a definition of *effectively complete* asset markets in a model of a single consumption good and two consumption periods. They defined asset markets as being effectively complete if every Pareto-efficient allocation can be attained through some trades of assets, and proved (Theorem 16.4.1) that the equilibrium allocations are Pareto-efficient in effectively complete asset markets. They then provided three examples, to be touched on later, for which asset markets are effectively complete and equilibrium allocations are easy to characterize. These examples shows that the notion of effectively complete asset market, restrictive as it may seem, deserves special attention thanks to its applicability to many important economic issues.

In this paper, we extend LeRoy and Werner's definition of effectively complete asset markets to the case of multiple goods and over multiple periods. Although the extension is straightforward and the class of economies with effectively complete asset markets is not very large, it admits several important applications. We then prove that, as in the case of the original definition of LeRoy and Werner (2001), if asset markets are effectively complete, then every equilibrium allocation is Pareto-efficient. This is our first welfare theorem in effectively complete asset markets.

In a number of special classes of economies, Pareto-efficient allocations are easy to characterize and asset markets are effectively complete. For such economies, the first welfare theorem in effectively complete asset markets can be used to characterize equilibrium allocations. The first application of the theorem, presented in Section 4, is to show that if the aggregate endowments are deterministic and constant over time, each consumer's endowments are generated by a portfolio of traded assets, and if all of them have the same discount factor, then each consumer's consumptions are also deterministic and constant over time. The second application, presented in Section 5, is the mutual fund theorem, which claims that if each consumer's endowments are generated by a portfolio of traded assets and all consumers have constant and equal relative risk aversion, and also equal discount factors, then each of them holds a fraction of aggregate endowments at equilibrium, and the fraction is deterministic and constant over time. The third application, presented in Section 6, is to extend the sunspot irrelevance theorem of Mas-Colell (1992) by showing that, in economies with no fundamental risk, if the aggregate endowments are constant over time, all consumers have the same discount factor, and if for each consumption good, there is a "consol," which always pays one unit of the good over the entire time span, then each consumer's consumptions are deterministic and constant over time. The fourth application, presented in Section 7, is to extend the no-retrade theorem in Markov economies of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003) to the case where the asset prices need not be time-invariant Markov processes.

There are, of course, special classes of effectively complete asset markets that we shall not analyze here. LeRoy and Werner (2001, Section 16.6) argued that if options of all exercise prices are traded,[1] then asset markets are effectively complete. In a dynamic model, Baptista (2003) showed that if American call and put options of all exercise prices are traded,[2] then asset markets are generically complete. He also showed by means of an example that more European options may be necessary to make asset markets complete than American options.

Another topic we shall not pursue in this paper is what can be termed as "asymptotically effectively complete" asset markets, that is, asset markets in which the equilibrium prices, consumptions, and utility levels are all close to their counterparts obtained in complete asset markets if there are sufficiently many, possibly infinitely many, periods and the consumers' discount factors are close to one. Levine and Zame (2002) asked under what conditions on asset markets they are or are not asymptotically effectively complete in models with infinitely many periods. They obtained three results. First, if there is only one consumption good and there is no aggregate risk, then short-lived riskless bonds are sufficient to make markets asymptotically effectively complete. Second, even when the aggregate endowments are risky, if there is only one consumption good, the shares of aggregate endowments across consumers are stochastically independent of aggregate endowments, short-lived riskless bonds are always available for trade, asset markets for aggregate endowments are complete, and if the consumers have constant and equal relative risk aversion, then asset markets are asymptotically effectively complete. Third, when there are more than one consumption goods, asset markets may not be asymptotically effectively complete even if there is no fundamental risk and short-lived riskless bonds are always available for trade.

Although we do not analyze asymptotic effective completeness, we do take up special classes of economies that are similar to those investigated by Levine and Zame (2002). In economies with no aggregate risk taken up in Section 4 and in economies with constant and equal relative risk aversion taken up in Section 5, our additional assumption, over those for the first two results of Levine and Zame (2002), is that each consumer's endowments are generated by a portfolio of traded assets, which guarantees that asset markets are effectively complete. In economies with no fundamental risk considered by Levine and Zame (2002), the (relative) spot prices fluctuate over time, and this fluctuation can be considered as being generated by sunspots. In economies with no fundamental risk taken up in Section 6, our additional assumption, over those for the third result of Levine and Zame (2002), is that for each consumption good there is a consol, which pays always one unit of the good over the entire time span. Our result in that section implies that the consols are sufficient to prevent sunspots from generating spot-price fluctuations. The results in this paper, therefore, clarify the assumptions needed to guarantee that the equilibrium allocations are Pareto-efficient even when there are only few periods or the consumers are fairly impatient.

This paper is organized as follows. Section 2 describes the setup for our analysis. Section 3 gives the definition of effectively complete asset markets and establishes the first welfare theorem. Section 4 provides the first application of effectively complete asset markets by

[1] More generally, if the payoff of the option of any exercise price can be obtained by trading assets.

[2] More generally, if the payoff of any such option can be obtained by trading assets.

establishing effective completeness in economies with no aggregate risk. Section 5 proves the mutual fund theorem. Section 6 extends the sunspot irrelevance theorem to our dynamic setting. Section 7 extends the no-retrade theorem. Section 8 sums up our analysis and suggests a direction of future research.

2 Setup and complete markets

There are $1+T$ periods, $t = 0, 1, \ldots, T$. There are S possible states of the world, $s = 1, 2, \ldots, S$ over the entire time span $\{0, 1, \ldots, T\}$. The gradual information revelation concerning the true state of the world is given by the filtration $(\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_T)$. We assume that $\mathscr{F}_0 = \{\varnothing, \{1, 2, \ldots, S\}\}$ and $\mathscr{F}_T$ coincides with the power set of $\{1, 2, \ldots, S\}$. Let P be a probability measure on $\{1, 2, \ldots, S\}$ such that $P(\{s\}) > 0$ for every s. For each positive integer n, we denote by X^n the set of all processes taking values in $\boldsymbol{R}^n$ over the time span $\{0, 1, \ldots, T\}$ that are adapted to the filtration $(\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_T)$. This is a linear space of finite dimension. Denote by X^n_+ the set of all processes taking values in $\boldsymbol{R}^n_+$, and by X^n_{++} the set of all processes in X^n taking values in $\boldsymbol{R}^n_{++}$.

There are L types of physically distinguished perishable goods, $\ell = 1, 2, \ldots, L$ on each period and state.

There are I consumers, $i = 1, 2, \ldots, I$. Their consumption sets are X^L_+, utility functions are $U_i : X^L_+ \to \boldsymbol{R}$, and initial endowments are $e^i = (e^i_0, e^i_t, \ldots, e^i_T) \in X^L$. We assume that the U_i are continuous and strongly monotone. We say that an allocation $(x^1, x^2, \ldots, x^I)$ of contingent commodities (consumption processes) is *feasible* if $x^i \in X^L_+$ for every i and $\sum_i x^i = \sum_i e^i$.

Definition 1 A feasible allocation $(x^1_*, x^2_*, \ldots, x^I_*)$ of consumption processes is *Pareto-superior* to another feasible allocation $(x^1, x^2, \ldots, x^I)$ if $U_i(x^i_*) \geq U_i(x^i)$ for every i and $U_i(x^i) > U_i(x^i_*)$ for some i. It is *Pareto-efficient* if no other feasible allocation is Pareto-superior to it.

A *contingent-commodity price process* is an element of X^L that represents the prices, at $t = 0$, for all the contingent commodities available over the entire time span. Under a contingent-commodity price process $\pi = (\pi_0, \pi_1, \ldots, \pi_T)$, consumer i can afford the consumption processes $x^i \in X^L_+$ that satisfy

$$E\left(\sum_{t=0}^{T} \pi_t \cdot \left(x^i_t - e^i_t\right)\right) \leq 0. \tag{1}$$

Definition 2 The pair of a feasible contingent-commodity allocation $(x^1_*, x^2_*, \ldots, x^I_*)$ and a contingent-commodity price process π is a *contingent-commodity market equilibrium* if for every i, $x^i = x^i_*$ maximizes $U_i(x^i)$ under the budget constraint (1).

Since the first welfare theorem is valid for a contingent-commodity market equilibrium, every contingent-commodity equilibrium allocation is Pareto-efficient.

Suppose now that there are J assets, $j = 1, 2, \ldots, J$, available for trade in the economy. Each asset j is characterized by its dividend process $d^j = (d^j_0, d^j_t, \ldots, d^j_T) \in X^L$. An *asset price process* is an element of X^J that represents the transition, expected by all consumers, of asset prices under uncertainty and over time. A *spot price process* is, just like a contingent-commodity price process, an element of X^L but it is interpreted as representing the transition, expected by all consumers, of prices for the L goods, for immediate consumption, under uncertainty and over time. A *trading plan* is an element of X^J that represents the transition, planned by a consumer, of portfolios of the J assets under uncertainty and over time.

Suppose consumer i employs a trading plan y^i under the asset price process q and a spot price process p. Define $d^{y^i} = (d^{y^i}_0, d^{y^i}_1, \ldots, d^{y^i}_T) \in X^1$ by

$$d^{y^i}_0 = -\sum_j q^j_0 y^{ji}_0, \tag{2}$$

$$d^{y^i}_t = \sum_j y^{ji}_{t-1}(p_t \cdot d^j_t) - \sum_j q^j_t (y^{ji}_t - y^{ji}_{t-1}) \text{ for every } t \geq 1, \tag{3}$$

where $q = (q_0, q_1, \ldots, q_T)$ with $q_t = (q^1_t, q^2_t, \ldots, q^J_t)$ for each t, $p = (p_0, p_1, \ldots, p_T)$, and $y^i = (y^i_0, y^i_1, \ldots, y^i_T)$ with $y^i_t = (y^{1i}_t, y^{2i}_t, \ldots, y^{Ji}_t)$ for each t. Then he can finance any consumption process $x^i \in X^L_+$ that satisfies

$$p_t \cdot \left(x^i_t - e^i_t\right) \leq d^{y^i}_t \tag{4}$$

for every $t \geq 0$, where $x^i = (x^i_0, x^i_1, \ldots, x^i_T)$.

An allocation $(y^1, y^2, \ldots, y^I)$ of trading plans is *feasible* if $\sum_i y^i = 0$.

Definition 3 The collection of a feasible allocation $(x^1_*, x^2_*, \ldots, x^I_*)$ of consumption processes, a feasible allocation $(y^1_*, y^2_*, \ldots, y^I_*)$ of trading plans, an asset price process q, and a spot price process p is an *asset market equilibrium* if for every i, $(x^i, y^i) = (x^i_*, y^i_*)$ maximizes $U_i(x^i)$ under the budget constraint (4) for every $t \geq 0$.

Since the U_i are strongly monotone, $p \in X^L_{++}$ and the weak inequality in (4) holds as an equality.

If $L = 1$, then, by replacing q_t by $(1/p_t)q_t$, we can assume that $p_t = 1$ for every t. This convention will be used throughout this paper without further notice. Even when $L \geq 2$, for every $\lambda \in X^1_{++}$, the collection of $(x^1_*, x^2_*, \ldots, x^I_*)$, $(y^1_*, y^2_*, \ldots, y^I_*)$, an asset price process $(\lambda_0 q_0, \lambda_1 q_1, \ldots, \lambda_T q_T)$, and a spot price process $(\lambda_0 p_0, \lambda_1 p_1, \ldots, \lambda_T p_T)$ is an asset market equilibrium. This property of an asset market equilibrium is often referred to as the *numéraire invariance* in the finance literature.

A trading plan y^i is called an *arbitrage* under an asset price process q and a spot price process $p \in X^L$ if $d^{y^i} = (d^{y^i}_0, d^{y^i}_1, \ldots, d^{y^i}_T) \in X^1_+ \setminus \{0\}$, where the $d^{y^i}_t$ are defined by (2) and (3). An arbitrage is a trading strategy that neither requires investment nor incurs obligation, and yet generates positive revenues on some period with positive probability. If there is no arbitrage under an asset price process q and a spot price process p, we say that q and p are *arbitrage-free.* Since the utility functions U_i are strongly monotone, if q and p are equilibrium price processes, then they are arbitrage-free. It is well known that whenever q and p are arbitrage-free, then there is a $\lambda \in X^1_{++}$ such that

$$E\left(\sum_{t=0}^{T} \lambda_t d_t^{y^i}\right) = 0 \tag{5}$$

for every $y^i \in X^J$. For any such λ, it can be shown that

$$\sum_j q_t^j y_t^{ji} = E_t\left(\sum_{\tau=t+1}^{T} \frac{\lambda_\tau}{\lambda_t} d_\tau^{y^i}\right) \tag{6}$$

for every $y^i \in X^J$ and $t \leq T-1$.

Denote by $M(q,p)$ the set of all $z^i \in X^L$ for which there is a $y^i \in X^J$ such that $p_t \cdot z_t^i = d_t^{y^i}$ for every $t \geq 1$. This is a linear subspace of X^L, often called the *market span.* Complete asset markets are then defined in terms of $M(q,p)$

Definition 4 Asset markets are *complete* under the equilibrium asset price process q and spot price process p if $M(q,p) = X^J$.

A contingent-commodity price process $\pi \in X^L$ is said to *support* the contingent-commodity allocation $(x_*^1, x_*^2, \ldots, x_*^I)$ on $M(q,p)$ if for every i and $x^i \in X_+^L$, $E\left(\sum_{t=0}^T \pi_t \cdot \left(x_t^i - x_{*t}^i\right)\right) > 0$ whenever $U_i(x^i) > U_i(x_*^i)$ and $x^i - x_*^i \in M(q,p)$. Then, for every $\lambda \in X^1_{++}$ satisfying (5), $(\lambda_0 p_0, \lambda_1 p_1, \ldots, \lambda_T p_T)$ supports $(x_*^1, x_*^2, \ldots, x_*^I)$ on $M(q,p)$. Indeed, every x^i that satisfies $E\left(\sum_{t=0}^T \lambda_t p_t \cdot \left(x_t^i - x_{*t}^i\right)\right) \leq 0$ also satisfies $E\left(\sum_{t=0}^T \lambda_t p_t \cdot \left(x_t^i - e_t^i\right)\right) \leq 0$. By (6),

$$E\left(\sum_{t=0}^{T} \lambda_t p_t \cdot \left(x_t^i - e_t^i\right)\right) = \lambda_0 \left(p_0 \cdot \left(x_0^i - e_0^i\right) + \sum_j q_0^j y_0^{ji}\right). \tag{7}$$

for some trading plan $y^i \in X^J$. Since $E\left(\sum_{t=0}^T \lambda_t p_t \cdot \left(x_t^i - e_t^i\right)\right) \leq 0$, $p_0 \cdot \left(x_0^i - e_0^i\right) + \sum_j q_0^j y_0^{ji} \leq 0$. This means that (x^i, y^i) satisfies the budget constraint of the asset market equilibrium. Hence $U_i(x^i) \leq U_i(x^i)$.

The following theorem is (one direction of) the well known equivalence between asset market equilibria in complete markets and contingent-commodity market equilibria. Since the first welfare theorem holds for contingent-commodity market equilibria, the equivalence implies that the asset market equilibrium allocations in complete markets are Pareto efficient.

Theorem 1 *If the collection of an allocation $(x_*^1, x_*^2, \ldots, x_*^I)$ of contingent commodities, an allocation $(y_*^1, y_*^2, \ldots, y_*^I)$ of trading plans, an asset price process q, and a spot price process p is an asset market equilibrium, and if asset markets are complete under q and p, then $(x_*^1, x_*^2, \ldots, x_*^I)$ is Pareto-efficient. Moreover, there exists a $\lambda \in X^1_{++}$ such that (6) holds and* *the pair of* $(x_*^1, x_*^2, \ldots, x_*^I)$ *and a contingent-commodity price process* $(\lambda_0 p_0, \lambda_1 p_1, \ldots, \lambda_T p_T)$ *is a contingent-commodity market equilibrium.*

3 Effectively complete markets and the first welfare theorem

We now give the definition of effectively complete asset markets.

Definition 5 Asset markets are *effectively complete* under the equilibrium asset price process q and spot price process p if for every Pareto-efficient allocation $(x^1, x^2, \ldots, x^I)$ of contingent commodities that is Pareto-superior to the equilibrium contingent-commodity allocation and for every i, $x^i - e^i \in M(q, p)$.

According to this definition, asset markets are effectively complete if at every Pareto-efficient allocation that is Pareto-superior to the equilibrium allocation,[3] every consumer can finance his consumption process by trading goods and assets under the equilibrium asset and spot price processes.[4]

If the equilibrium contingent-commodity allocation is Pareto-efficient, then asset markets are, trivially, effectively complete under the equilibrium asset and spot price process. It also follows directly from the definition that complete asset markets are effectively complete.

A sufficient condition for asset markets to be effectively complete is that every Pareto-efficient allocation be attainable by the buy-and-hold strategies, even without trading on spot markets at all. It is a simple and yet useful condition, because it is independent of the equilibrium price processes. Indeed, it is satisfied by all the examples of effectively complete asset markets in the next four sections.

Lemma 1 *Asset markets are effectively complete if for every individually rational and Pareto-efficient allocation* $(x^1, x^2, \ldots, x^I)$ *and for every* i*, there exists a* $\theta^i \in \boldsymbol{R}^J$ *such that*

$$x_t^i - e_t^i = \sum_j \theta^{ji} d_t^j \tag{8}$$

for every $t \geq 1$.

Proof of Lemma 1 Let the collection of an allocation $(x_*^1, x_*^2, \ldots, x_*^I)$ of consumption processes, an allocation $(y_*^1, y_*^2, \ldots, y_*^I)$ of trading plans, an asset price process q, and a spot price process p be an asset market equilibrium. Let $(x^1, x^2, \ldots, x^I)$ be a Pareto-efficient allocation that is Pareto-superior to $(x_*^1, x_*^2, \ldots, x_*^I)$. Since $(x_*^1, x_*^2, \ldots, x_*^I)$ is individually

[3] Since the utility functions are not assumed to be strictly quasi-concave, there may be multiple equilibrium contingent-commodity allocations, given the asset price process q and spot price process p. Yet the requirement that $(x^1, x^2, \ldots, x^I)$ be Pareto-superior to the equilibrium allocation does not depend on its choice, because every consumer is indifferent between his own consumption processes of any two equilibrium allocations of the same equilibrium price processes.

[4] I am grateful to the anonymous referee for pointing out that the original definition of effective completeness, which is essentially the same as the condition in Lemma 1, is too demanding and suggesting a less demanding definition similar to this one.

rational, so is $(x^1, x^2, \ldots, x^I)$. Thus, for every i, there exists a $\theta^i \in \boldsymbol{R}^J$ for which (8) holds. Define $y^i \in X^J$ by letting $y^i_t = \theta^i$ for every $t \geq 0$. Since $y^i_t - y^i_{t-1} = 0$, (8) implies that

$$d_t^{y^i} = \sum_j \theta^{ji} \left(p_t \cdot d_t^j \right) = p_t \cdot \left(\sum_j \theta^{ji} d_t^j \right) = p_t \cdot \left(x_t^i - e_t^i \right)$$

for every $t \geq 1$. Thus $x^i - e^i \in M(q, p)$ and asset markets are effectively complete. ///

Although effective completeness is less demanding than completeness, the first welfare theorem is still valid in effectively complete asset markets. In other words, effective completeness of asset markets is not only necessary but also sufficient for Pareto-efficiency of the equilibrium contingent-commodity allocations. It is because of this theorem that we have dubbed the property stated in Definition 5 "effective completeness".

Theorem 2 *If the collection of an allocation* $(x^1_*, x^2_*, \ldots, x^I_*)$ *of contingent commodities, an allocation* $(y^1_*, y^2_*, \ldots, y^I_*)$ *of trading plans, an asset price process* q*, and a spot price process* p *is an asset market equilibrium, and if asset markets are effectively complete under* q *and* p*, then* $(x^1_*, x^2_*, \ldots, x^I_*)$ *is Pareto-efficient.*

Proof of Theorem 2 Suppose that the collection of a contingent-commodity allocation $(x^1_*, x^2_*, \ldots, x^I_*)$, an allocation $(y^1_*, y^2_*, \ldots, y^I_*)$ of trading plans, an asset price process q, and a spot price process p is an asset market equilibrium, and that a feasible contingent-commodity allocation $(x^1, x^2, \ldots, x^I)$ is Pareto-superior to $(x^1_*, x^2_*, \ldots, x^I_*)$.

As shown in LeRoy and Werner (2001, Proposition 16.3.2), since the consumption sets are closed and bounded from below and the utility functions are continuous, there is a Pareto-efficient allocation that is Pareto-superior to $(x^1_*, x^2_*, \ldots, x^I_*)$. Without loss of generality, therefore, we can assume that $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient. By effective completeness, for each $i \geq 2$, there exists a $y^i \in X^J$ such that (4) is satisfied on each period $t \geq 1$. Let $y^1 = -\sum_{i \geq 2} y^i \in X^J$, then $(y^1, y^2, \ldots, y^I)$ is a feasible allocation of trading plans and (4) is satisfied for $i = 1$ on each period $t \geq 1$. Thus, for every i with $U_i(x^i) > U_i(x^i_*)$, (4) fails to hold on period 0, that is,

$$p_0 \cdot \left(x_0^i - e_0^i \right) > -\sum_j q_0^j y_0^{ji}. \tag{9}$$

For every i with $U_i(x^i) = U_i(x^i_*)$, since U_i is strongly monotone,

$$p_0 \cdot \left(x_0^i - e_0^i \right) \geq -\sum_j q_0^j y_0^{ji}. \tag{10}$$

Summing up (9) and (10) over i and using the feasibility constraints, we obtain

$$0 > -\sum_j q_0^j \left(\sum_i y_0^{ji} \right) = 0,$$

which is a contradiction. Thus $(x^1_*, x^2_*, \ldots, x^I_*)$ is Pareto-efficient. ///

Unlike completeness, effective completeness falls short of guaranteeing that the equilibrium allocation is also obtained at some contingent-commodity market equilibrium. The following example is constructed by modifying the utility functions in Example 16.4.4 of LeRoy and Werner (2001) to satisfy strong monotonicity.

Example 1 Let $T = 1$, $S = 2$, $L = 1$, and $I = 2$. Define consumer 1's initial endowment $e^1 = (e^1_0, e^1_1) = (e^1_0, (e^1_1(1), e^1_1(2)))$ by letting $e^1_0 = 1$, $e^1_1(1) = 0$, and $e^1_1(2) = 1$. Define consumer 2's initial endowment $e^2 = (e^2_0, e^2_1) = (e^2_0, (e^2_1(1), e^2_1(2)))$ by letting $e^2_0 = 1$, $e^2_1(1) = 1$, and $e^2_1(2) = 0$. Let $a \in (1/2, 1)$. Define consumer 1's utility function U_1 by letting

$$U_1(x^1) = x^1_0 + ax^1_1(1) + (1-a)x^1_1(2)$$

for every $x^1 = (x^1_0, (x^1_1(1), x^1_1(2))) \in X^1_+$. Define consumer 2's utility function U_2 by letting

$$U_2(x^2) = x^2_0 + (1-a)x^2_1(1) + ax^2_1(2)$$

for every $x^2 = (x^2_0, (x^2_1(1), x^2_1(2))) \in X^1_+$.

Let $J = 1$ and define the dividend process $d^1 = (d^1_0, (d^1_1(1), d^1_1(2)))$ by letting $d^1_0 = 0$ $d^1_1(1) = 1$ and $d^1_1(2) = -1$.

Proposition 1 *In Example 1:*

1. *For every $b \in (-1, 1)$, the consumption allocation (x^1_*, x^2_*) defined by*

$$x^1_* = (1-b, (1,0)) \text{ and } x^2_* = (1+b, (0,1)). \tag{11}$$

   is Pareto-efficient, and it has an essentially unique supporting contingent-commodity price process, given by $\pi = (1, (a, a))$.

2. *There is a unique contingent-commodity market equilibrium $((x^1_*, x^2_*), \pi)$, which is given by $b = 0$ in (11).*

3. *For every asset market equilibrium $((x^1_*, x^2_*), (y^1_*, y^2_*), q)$, there is a $b \in (1-2a, 2a-1)$ such that (x^1_*, x^2_*) is given by (11) and*

$$\begin{aligned} y^1_* &= (1, (0,0)), \\ y^2_* &= (-1, (0,0)), \\ q &= (b, (0,0)). \end{aligned}$$

   Asset markets are effectively complete under every equilibrium asset price.

4. *The contingent-commodity allocations of some asset market equilibria are not the same as the contingent-commodity allocation of the contingent-commodity market equilibrium.*

Instead of providing a formal proof of this proposition, we give some intuitive account on it. Part 1 characterizes some, but not all, Pareto-efficient allocations. It shows that due to the difference in marginal utilities, a and $1-a$, from consumptions in the two states on time 1, it is efficient for each consumer to consume only in one of the two states. Since the utility functions are strongly monotone, for each $b \in [0,1]$, the contingent-commodity allocations (x^1_*, x^2_*) defined by

$$x^1_* = (0, (1-b, 0)) \text{ and } x^2_* = (2, (b, 1)).$$

and also by

$$x^1_* = (2, (1, b)) \text{ and } x^2_* = (0, (0, 1-b)).$$

are Pareto-efficient. These allocations, however, are neither individually rational nor attainable by trading the asset. Part 2 tells us which one of the Pareto-efficient allocations is an equilibrium allocation. Part 3 shows that there is a continuum of asset market equilibria, all of which share the same period-1 contingent-commodity allocations. Since asset markets are incomplete with the degree of incompleteness $S - J = 1$, for each equilibrium contingent-commodity allocation, there is one degree of freedom of the contingent-commodity price processes that support it on the market span $M(q,p)$. Yet, there is one more degree of freedom of the supporting price processes, because the equilibrium contingent-commodity allocations lie on the boundary of each consumer's consumption set X^1_+. Indeed, every $\pi = (\pi_0, (\pi^1_1, \pi^2_1)) \in X^1_{++}$ satisfying $\pi_0 = 1$ and $1 - 2a < \pi^1_1 - \pi^2_1 < 2a - 1$ supports (x^1_*, x^2_*) on $M(q,p)$. As observed by Elul (1999), this additional degree of freedom is the driving force behind Part 4: unless $b = 0$ in (11), the asset market equilibrium allocation is not a contingent-commodity equilibrium allocation.

We now formalize the last point of the previous paragraph. Let the collection of the contingent-commodity allocation $(x^1_*, x^2_*, \ldots, x^I_*)$, the trading-plan allocation $(y^1_*, y^2_*, \ldots, y^I_*)$, the asset price process q, and the spot price process p be an asset market equilibrium. For every $\lambda \in X^1_{++}$, if (5) holds, then $(\lambda_0 p_0, \lambda_1 p_1, \ldots, \lambda_T p_T)$ supports $(x^1_*, x^2_*, \ldots, x^I_*)$ on $M(q,p)$. As Example 1 shows, however, there may be other contingent-commodity price processes that support $(x^1_*, x^2_*, \ldots, x^I_*)$ on $M(q,p)$. The following theorem shows that if there is essentially no other contingent-commodity price process, then the equilibrium allocation is the same as the equilibrium allocation of some contingent-commodity market equilibrium.

Theorem 3 *Suppose that U_i is quasi-concave for every i. Let the collection of the contingent-commodity allocation $(x^1_*, x^2_*, \ldots, x^I_*)$, the trading plan allocation $(y^1_*, y^2_*, \ldots, y^I_*)$, the asset price process q, and the spot price process p be an asset market equilibrium. Suppose that asset markets are effectively complete under q and p. Suppose also that for every $\pi \in X^L$, if*

π supports $(x^1_*, x^2_*, \ldots, x^I_*)$ on $M(q,p)$, then there exists a $\lambda \in X^1_{++}$ such that (5) holds and

$$E\left(\sum_{t=0}^{T} \lambda_t p_t \cdot z_t\right) = E\left(\sum_{t=0}^{T} \pi_t \cdot z_t\right) \tag{12}$$

for every $z \in M(q,p)$. Then there exists a $\pi \in X^1$ such that the pair of $(x^1_*, x^2_*, \ldots, x^I_*)$ and π is a contingent-commodity market equilibrium.

Proof of Theorem 3 Since the U_i are strongly monotone and quasi-concave, there exists a $\pi \in X^L_{++}$ that supports $(x^1_*, x^2_*, \ldots, x^I_*)$ on X^L.[5] It suffices to show that for every i,

$$E\left(\sum_{t=0}^{T} \pi_t \cdot \left(x^i_{*t} - e^i_t\right)\right) \le 0. \tag{13}$$

Indeed, since $\pi \in X^L_{++}$ supports $(x^1_*, x^2_*, \ldots, x^I_*)$ on X^L, it does so on $M(q,p)$ in particular. By assumption, there exists a $\lambda \in X^1_{++}$ for which (5) and (12) hold. Since $x^i_{*t} - e^i_t \in M(q,p)$,

$$E\left(\sum_{t=0}^{T} \pi_t \cdot \left(x^i_{*t} - e^i_t\right)\right) = E\left(\sum_{t=0}^{T} \lambda_t p_t \cdot \left(x^i_{*t} - e^i_t\right)\right) = \lambda_0 \left(p_0 \cdot \left(x^i_{*0} - e^i_0\right) + \sum_j q^j_0 y^{ji}_0\right) \le 0,$$

where the second equality following from (7) and the last inequality follows from the budget constraint of the asset market equilibrium. ///

The crucial assumption of this theorem is that for every $\pi \in X^L$ supporting $(x^1_*, x^2_*, \ldots, x^I_*)$ on $M(q,p)$, there exists a $\lambda \in X^1_{++}$ such that (5) and (12) hold. Since there is no arbitrage under equilibrium price processes, there always exists a $\lambda \in X^1_{++}$ such that (5) holds. Thus, there always exists a contingent-commodity price process of the form $(\lambda_0 p_0, \lambda_1 p_1, \ldots, \lambda_T p_T)$ that supports $(x^1_*, x^2_*, \ldots, x^I_*)$ on $M(q,p)$. Hence, the assumption is met if the supporting contingent-commodity price process is essentially unique on $M(q,p)$. Stated formally, the assumption is met if there exists a $k \in \boldsymbol{R}_{++}$ such that

$$E\left(\sum_{t=0}^{T} \pi_t \cdot z_t\right) = kE\left(\sum_{t=0}^{T} \varphi_t \cdot z_t\right)$$

for every $z \in M(q,p)$ whenever both $\pi \in X^L_{++}$ and $\varphi \in X^L_{++}$ support $(x^1_*, x^2_*, \ldots, x^I_*)$ on $M(q,p)$. As shown by Elul (1999), this essential uniqueness is obtained if there is a consumer i such that $x^i_* \in X^L_{++}$ and U_i is differentiable. More generally, the assumption is met if there is a basis of $M(q,p)$ of the form $(z^0, z^1, \ldots, z^N)$ such that for every $n = 1, 2, \ldots, N$, there

[5] The standard second welfare theorem only claims that $\pi \in X^L_{++}$ supports $(x^1_*, x^2_*, \ldots, x^I_*)$ on X^L in the weak sense, that is, $E\left(\sum_{t=0}^{T} \pi_t \cdot x^i_t\right) \ge E\left(\sum_{t=0}^{T} \pi_t \cdot x^i_{*t}\right)$ whenever $U_i(x^i) > U_i(x^i_*)$. However, since the consumption sets are X^L_+ and the utility functions U_i are strongly monotone in our setting, the strict inequality, $E\left(\sum_{t=0}^{T} \pi_t \cdot x^i_t\right) > E\left(\sum_{t=0}^{T} \pi_t \cdot x^i_{*t}\right)$, is obtained.

exists a consumer i such that $x_*^i + a^0 z^1 + a^n z^n \in X_+^L$ for every $(a^0, a^n) \in \boldsymbol{R}^2$ sufficiently close to $(0,0)$, and if U_i is differentiable at x_*^i on the plane spanned by z^0 and z^n. In this case, the relative price between z^0 and z^n embedded in any contingent-commodity price process supporting $(x_*^1, x_*^2, \ldots, x_*^I)$ on $M(q,p)$ is uniquely determined by consumer i's marginal rate of substitution between them. Since this is true for all n, the contingent-commodity price process supporting $(x_*^1, x_*^2, \ldots, x_*^I)$ on $M(q,p)$ is uniquely determined on $M(q,p)$, and hence the assumption of Theorem 3 is met.

4 No aggregate risk

In this section, we give our first application of Theorem 2, the first welfare theorem in effectively complete markets. It is that asset markets are effectively complete when all the consumers' initial endowments are, in fact, endowments in the traded assets, the aggregate endowments deterministic and constant over time, and the consumers have the same discount factor.

Formally, we assume that $L = 1$, that is, there is only one good in each state and on each period. Assume also that there is a $\delta > 0$ such that

$$U_i(x^i) = E\left(\sum_{t=0}^{T} \delta^t u_i\left(x_t^i\right)\right) = \sum_{t=0}^{T}\sum_{s=1}^{S} \delta^t P(\{s\}) u_i(x_t^i(s))$$

for each i, where $u_i : \boldsymbol{R}_+ \to \boldsymbol{R}$ is continuous, strongly monotone, and strictly concave.

We also assume that for each i and j, there is a $\theta^{ji} \geq 0$ such that $e^i = \sum_{j=1}^{J} \theta^{ji} d^j$ for every i. We assume, without loss of generality, that $\sum_{i=1}^{I} \theta^{ji} = 1$ for every j. Then the aggregate endowment process e is equal to $\sum_{j=1}^{J} d^j$. We assume that e is deterministic and constant over time.

We start with characterizing the Pareto-efficient allocations in this economy. The following lemma shows that all consumers' consumptions are deterministic and constant over time at every Pareto-efficient allocation.

Lemma 2 *If a feasible contingent-commodity allocation $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient, then x^i is deterministic and constant over time for every i.*

Proof of Lemma 2 For any feasible allocation $(x^1, x^2, \ldots, x^I)$, define another allocation $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^I)$ by $\hat{x}_t^i = \bar{x}^i$ for every i and t with probability one, where

$$\bar{x}^i = \sum_{t=0}^{T}\sum_{s=1}^{S} \frac{\delta^t P(\{s\})}{1 + \delta + \cdots + \delta^T} x_t^i(s) \in \boldsymbol{R}_+.$$

Then, for every i, $\hat{x}^i$ is constant and deterministic. Also, $(\hat{x}^1, \hat{x}^2, \dots, \hat{x}^I)$ is feasible because

$$\sum_i \hat{x}^i_t(s) = \sum_i \sum_{t'} \sum_{s'} \frac{\delta^{t'} P(\{s'\})}{1+\delta+\cdots+\delta^T} x^i_{t'}(s') = \sum_{t'} \sum_{s'} \frac{\delta^{t'} P(\{s'\})}{1+\delta+\cdots+\delta^T} \sum_i x^i_{t'}(s')$$
$$= \sum_{t'} \sum_{s'} \frac{\delta^{t'} P(\{s'\})}{1+\delta+\cdots+\delta^T} e_{t'}(s') = e_t(s),$$

where the last equality follows from the assumption that e is deterministic and constant over time. Since u_i is strictly concave,

$$U_i(x^i) = (1+\delta+\cdots+\delta^T) \sum_{t=0}^{T} \sum_{s=1}^{S} \frac{\delta^t P(\{s\})}{1+\delta+\cdots+\delta^T} u_i(x^i_t(s)) \le (1+\delta+\cdots+\delta^T) \sum_{t=0}^{T} \delta^t u_i(\bar{x}^i_t) = U_i(\hat{x}^i),$$

where the inequality holds as a strict inequality unless $\hat{x}^i = x^i$. Hence, $(\hat{x}^1, \hat{x}^2, \dots, \hat{x}^I)$ is Pareto-superior to $(x^1, x^2, \dots, x^I)$ unless $x^i = \hat{x}^i$, that is, x^i is deterministic and constant over time, for every i. Therefore, if $(x^1, x^2, \dots, x^I)$ is Pareto efficient, then x^i is constant and deterministic for every i. ///

Next, we prove that the above assumptions are sufficient to guarantee that asset markets are effectively complete.

Lemma 3 *Asset markets are effectively complete.*

Proof of Lemma 3 Let $(x^1, x^2, \dots, x^I)$ be a Pareto-efficient allocation. Then, by Lemma 2, there exists a $\zeta^i \in [0,1]$ such that $x^i = \zeta^i e$. Thus, $x^i_t - e^i_t = \sum_j \left(\zeta^i - \theta^{ji}\right) d^j_t$ for every $t \ge 1$. By Lemma 1, this implies that asset markets are effectively complete. ///

The following theorem generalizes the analysis of Section 16.5 of LeRoy and Werner (2001) to the case of multiple periods.

Theorem 4 *Every equilibrium allocation is Pareto-efficient and all consumers' consumption are deterministic and constant at every equilibrium allocation.*

5 Mutual fund theorem

In this section, we show that when all the consumers' initial endowments are, in fact, endowments in the traded assets, and the consumers have the same discount factor and the same coefficient of constant relative risk aversion, then asset markets are effectively complete.

Formally, we assume that $L = 1$ and there are a $\delta > 0$ and a $\gamma > 0$ such that

$$U_i(x^i) = E\left(\sum_{t=0}^{T} \delta^t u_i\left(x^i_t\right)\right)$$

for all i, where $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ is defined by

$$u(z) = \begin{cases} \ln z & \text{if } \gamma = 1, \\ \dfrac{z^{1-\gamma} - 1}{1 - \gamma} & \text{otherwise,} \end{cases}$$

for every $z \in \boldsymbol{R}_{++}$.[6] This means that all consumers have constant relative risk aversion equal to γ.

We also assume that for each i and j, there is a $\theta^{ji} \geq 0$ such that $e^i = \sum_{j=1}^{J} \theta^{ji} d^j$ for every i. We assume, without loss of generality, that $\sum_{i=1}^{I} \theta^{ji} = 1$ for every j. Then the aggregate endowment process e is equal to $\sum_{j=1}^{J} d^j$. Unlike Section 4, we do not assume that e is deterministic and constant over time.

The following lemma is known as the mutual fund theorem. It differs from its counterpart in a static setting in that the mutual fund property holds not only across states but also over time. Its validity follows from the assumption that all consumers have the same discount factor. We omit the proof.

Lemma 4 *If a feasible contingent-commodity allocation $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient, then for every i, there exists a $\zeta^i \in (0, 1)$ such that $x^i = \zeta^i e$.*

Just as Lemma 3, we can prove that the above assumptions are sufficient to guarantee that asset markets are effectively complete. We thus obtain the following theorem, which generalizes the analysis of Section 16.7 of LeRoy and Werner (2001) to the case of multiple periods.

Theorem 5 *Every equilibrium allocation is Pareto-efficient. If $(x^1, x^2, \ldots, x^I)$ is an equilibrium allocation, then for every i, there exists a $\zeta^i \in (0, 1)$ such that $x^i = \zeta^i e$.*

6 Sunspot irrelevance

In this section, we extend the sunspot-irrelevance theorem of Mas-Colell (1992) to the dynamic setting.

Unlike the previous two sections, we consider the case in which the number L of consumption goods may be greater than one. We assume that there is a $\delta > 0$ such that

$$U_i(x^i) = E\left(\sum_{t=0}^{T} \delta^t u_i\left(x_t^i\right)\right)$$

[6] To be precise, U_i is then defined on L^1_{++} rather than L^1_{+}, contrary to the assumption stated in Section 2. But this pauses no need to modify our analysis, because the Pareto-efficient allocations that we look into in conjunction with effective completeness are those which Pareto-dominate the equilibrium allocations; and they are necessarily in some compact subset of L^1_{++}. For a similar reason, we could extend our analysis to the case where the slopes of the risk tolerance (the reciprocal of absolute risk aversion) of all consumers are constant and equal, if we assume that consols (paying a deterministic and constant dividend from period 1 onwards) are traded. The details of such an extension are stated in Section 16.7 of LeRoy and Werner (2001).

for each i, where $u_i : \boldsymbol{R}^L_+ \to \boldsymbol{R}$ is continuous, strongly monotone, and strictly concave.

We also assume that for each i, the endowment process e^i is deterministic and constant over time. Hence, the states are irrelevant to utility functions and initial endowments, and thus called *sunspot* states. Of course, there is an asset market equilibrium of which the contingent-commodity allocation is sunspot-free. Mas-Colell (1992) showed, in the case of $T = 1$ and there is no consumption on period 0, that if there are not sufficiently many assets available for trade, then there may be an asset market equilibrium of which the contingent-commodity allocation depends on sunspots and its realizations are different from the sunspot-free equilibrium allocations.

We start with characterizing the Pareto-efficient allocations in this economy. The following lemma shows that they are sunspot-free. As it can be proved in the same way as Lemma 2, we omit the proof.

Lemma 5 *If a feasible contingent-commodity allocation* $(x^1, x^2, \dots, x^I)$ *is Pareto-efficient, then* x^i *is deterministic and constant over time for every* i.

Next, we show that asset markets are effectively complete if for each of the L consumption goods, there is a "consol," which always pays one unit of the good over the entire time span. Since the number L of consumption goods may well be less than the number of the events that can arise in the subsequent period after some event (except for those obtained at on the terminal period), these effectively complete asset markets may not be complete.

Lemma 6 *Asset markets are effectively complete if for every good* ℓ *there is an asset* j *such that*

$$d^j_t = (0, \dots, 0, \underbrace{1}_{\ell-th}, 0, \dots, 0) = (\ell\text{-}th\ unit\ vector) \in \boldsymbol{R}^L$$

with probability one, for every $t \geq 1$.

Proof of Lemma 6 If $(x^1, x^2, \dots, x^I)$ is an efficient allocation, then, by Lemma 5, for each i there exists a $z^i \in \boldsymbol{R}^L$ such that $x^i_t - e^i_t = z^i$ with probability one for every $t \geq 1$. By assumption, for every i, there is a portfolio $\theta^i \in \boldsymbol{R}^J$ such that $z^i = \sum_j \theta^{ji} d^{sj}_t$ with probability one, for every $t \geq 1$. Thus $\sum_j \theta^{ji} d^j_t = x^i_t - e^i_t$ for every $t \geq 1$. By Lemma 1, this implies that asset markets are effectively complete. ///

Under the assumption of Lemma 5, asset markets are effectively complete and, by Theorem 2, the equilibrium contingent-commodity allocations are Pareto-efficient and, by Lemma 6, the consumptions are deterministic and constant over time. We have thereby extended Mas-Colell's (1992) theorem to the dynamic setting.

Theorem 6 *If for every good* ℓ *there is an asset* j *such that* d^{sj}_t *is equal to the* ℓ*-th unit vector with probability one for every* $t \geq 1$*, then every equilibrium allocation is Pareto-efficient and each consumer's consumptions are deterministic and constant over time.*

We have mentioned that if the number of assets is less than the number of the events that can arise in the subsequent period after some event, then asset markets must necessarily be incomplete but the equilibrium allocations are Pareto-efficient. It may even be the case that these assets turn out to be redundant under the equilibrium prices, rendering asset markets incomplete regardless of whether J is larger or smaller than the number of the events that can arise in the subsequent period, while retaining the Pareto efficiency of the equilibrium allocations.

To see this point more formally, assume that $T = 1$ and for every $j \leq L$, the j-th asset pays out one unit of good j with probability one, and that the collection of a feasible contingent-commodity allocation $(x^1, x^2, \ldots, x^I)$, a feasible allocation $(y^1, y^2, \ldots, y^I)$ of trading plans, an asset price process q, and a spot price process p is an asset market equilibrium. Then $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient and x^i is deterministic and constant over time for every i. Define another spot price process $\hat{p} = (\hat{p}_0, \hat{p}_1)$ by letting $\hat{p}_0 = p_0$ and $\hat{p}_1$ coincide with the first L coordinates of q_0 with probability one.[7] Then $\hat{p}_1$ is sunspot-free. Define another feasible allocation $(\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^I)$ of trading plans by letting the first L coordinates of $\hat{y}^i_0$ coincide with $x^i_1 - e^i_1$ (which is sunspot-free) and the remaining $J - L$ coordinates equal to zero. Then the collection of $(x^1, x^2, \ldots, x^I)$, $(\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^I)$, q, and $\hat{p}$ is an asset market equilibrium.[8] This is because every sunspot-free consumption plan that can be attained under (q, p) can also be attained under $(q, \hat{p})$, and vice versa. In this latter equilibrium, the contingent-commodity allocation is the same as in the original equilibrium, but all but one assets are redundant, because the $\hat{p}_1 \cdot d^j_1$ are sunspot-free (that is, all assets are riskless bonds). This implies that asset markets are incomplete as long as $S \geq 2$, but the equilibrium allocation is Pareto-efficient.

7 No-retrade theorem

Our last application of effectively complete markets is the no-retrade theorem of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003).

We consider a Markov environment in which there are M states, $m = 1, 2, \ldots, M$, on each period and a single consumption good in each state. Let $\bar{m} \in \{1, 2, \ldots, M\}$ be the state on period 0, then the state space over the entire history is given by $S = \{\bar{m}\} \times M^T$.[9]

Define $\chi : S \times \{0, 1, \ldots, T\} \to M$ by $\chi(s, t) = s_t$, where $s = (s_0, s_1, \ldots, s_T) \in S$. Write χ_t for $\chi(\cdot, t) : S \to M$. Then χ_t maps each entire history to the state that arises on period t along the history. The filtration $(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_T)$ is defined in such a way that for every t, $\mathscr{F}_t$ is generated by the mapping $(\chi_0, \chi_1, \ldots, \chi_t) : S \to M^t$. That is, for every $s = (s_0, s_1, \ldots, s_T) \in S$ and $s' = (s'_0, s'_1, \ldots, s'_T) \in S$, s and s' belong to the same element of the partition $\mathscr{G}_t$ corresponding to $\mathscr{F}_t$ if and only if $s_{t'} = s'_{t'}$ for every $t' \leq t$.

[7] If, for example, there exists an i such that $x^i_0 \in \boldsymbol{R}^L_{++}$ and u^i is differentiable at x^i_0, then there exists a $\lambda \in X^1_{++}$ such that $\hat{p}_1 = \lambda_1 p_1$ and, as the subsequent argument will show, all but one assets are redundant even under p.

[8] If T were greater than one, then q would need to be multiplied by an appropriate state-price deflator.

[9] There is a slight abuse of notation, as S is a set in this section, while it used to be a positive integer up to the previous section. Little confusion will arise from this abuse of notation.

Assume that all consumers have the same discount factor $\delta > 0$ and have state-dependent expected utility functions

$$U_i(x^i) = E\left(\sum_{t=0}^{T} \delta^t u_i(x_t^i, \chi_t)\right) = \sum_{(s,t)\in S\times\{0,1,\dots,T\}} \delta^t P(\{s\}) u_i(x_t^i(s), s_t),$$

where $u_i : \boldsymbol{R}_+ \times \{1, 2, \dots, M\} \to \boldsymbol{R}$ and $x^i = (x_0^i, x_1^i, \dots, x_T^i)$.

Assume that, for the initial endowment process $e^i = (e_0^i, e_1^i, \dots, e_T^i)$ of each consumer i, each e_t^i depends only on s_t (and not on t), that is, there is a $g_i : \{1, 2, \dots, M\} \to \boldsymbol{R}$ such that $e^i = g_i(\chi)$. Assume also that, for the dividend process $d^j = (d_0^j, d_1^j, \dots, d_T^j)$ of each asset j, each d_t^j depends only on s_t (and not on t), that is, there is an $h_i : \{1, 2, \dots, M\} \to \boldsymbol{R}$ such that $d^j = h_j(\chi)$.

This economy is in the Markov environment, as there are M states that recur over time and the utility functions, initial endowments, and dividend payouts depend only on the state on the period but not on the state on any earlier period. But, unlike the model of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003), the probability that state m occurs on period t may depend not only on the state that occurred on period $t-1$ but also on some earlier periods. Note also that we are assuming that there are only finitely many periods, while they assumed that there are infinitely many periods. Finally, all assets in our model are long lived (traded from period 0 onwards and dividends paid out until period T), while some assets in their model may be short-lived (traded just once and dividends paid out only on the next period). We exclude short-lived assets from our model for the sake of simplicity of exposition.

Just as in the previous sections, we start the analysis of the model with characterizing the Pareto-efficient allocations.

Lemma 7 *If an allocation $(x^1, x^2, \dots, x^I)$ is Pareto-efficient, then for every i, there exists an $f_i : \{1, 2, \dots, M\} \to \boldsymbol{R}$ such that $x^i - e^i = f_i(\chi)$.*

Proof of Lemma 7 Let $(x^1, x^2, \dots, x^I)$ be a feasible contingent-commodity allocation. For each m, define $r^m = \sum_{(s,t)\in\chi^{-1}(m)} \delta^t P(\{s\})$. Then, for each m and i, define

$$\bar{x}^{mi} = \sum_{(s,t)\in\chi^{-1}(m)} \frac{\delta^t P(\{s\})}{r^m} x_t^i(s).$$

Then define $\hat{x}^i = \left(\hat{x}_0^i, \hat{x}_1^i, \dots, \hat{x}_T^i\right)$ by letting $\hat{x}_t^i(s) = \bar{x}^{\chi(s,t)i}$ for every (s,t). Then the allocation

$(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^I)$ is feasible because

$$\begin{aligned}
\sum_{i=1}^{I} \hat{x}^i(s) = \sum_{i=1}^{I} \bar{x}^{\chi(s,t)i} &= \sum_{i=1}^{I} \sum_{(s',t') \in \chi^{-1}(\chi(s,t))} \frac{\delta^{t'} P(\{s'\})}{r^{\chi(s,t)}} x^i_{t'}(s') \\
&= \sum_{(s',t') \in \chi^{-1}(\chi(s,t))} \frac{\delta^{t'} P(\{s'\})}{r^{\chi(s,t)}} \sum_{i=1}^{I} x^i_{t'}(s') \\
&= \sum_{(s',t') \in \chi^{-1}(\chi(s,t))} \frac{\delta^{t'} P(\{s'\})}{r^{\chi(s,t)}} \sum_{i=1}^{I} e^i_{t'}(s') \\
&= \sum_{(s',t') \in \chi^{-1}(\chi(s,t))} \frac{\delta^{t'} P(\{s'\})}{r^{\chi(s,t)}} \sum_{i=1}^{I} g_i(\chi(s,t)) \\
&= \sum_{i=1}^{I} g_i(\chi(s,t)) = \sum_{i=1}^{I} e^i(s).
\end{aligned}$$

Since $u_i(\cdot, m)$ is strictly concave,

$$U_i(x^i) = \sum_{m=1}^{M} r^m \sum_{(s,t) \in \chi^{-1}(m)} \frac{\delta^t P(\{s\})}{r^m} u_i\left(x^i_t(s), m\right) \leq \sum_{m=1}^{M} r^m \sum_{(s,t) \in \chi^{-1}(m)} u_i\left(\hat{x}^i_t(s), m\right) = U_i(\hat{x}^i),$$

where the weak inequality holds as a strict inequality unless $x^i = \hat{x}^i$, that is, $x^i_t(s) = x^i_{t'}(s')$ whenever $\chi(s,t) = \chi(s',t')$ for every i. Thus $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^I)$ is Pareto-superior to $(x^1, x^2, \ldots, x^I)$ unless $x^i_t(s) = x^i_{t'}(s')$ whenever $\chi(s,t) = \chi(s',t')$ for every i. Therefore, if $(x^1, x^2, \ldots, x^I)$ is Pareto-efficient, then $x^i_t(s) = x^i_{t'}(s')$ whenever $\chi(s,t) = \chi(s',t')$ for every i. This means that for every i, there is a $\hat{f}_i : \{1, 2, \ldots, M\} \to \boldsymbol{R}$ such that $x^i = \hat{f}_i(\chi)$. The proof is completed by taking $f_i = \hat{f}_i - g_i$. ///

To state a sufficient condition for effectively complete asset markets, write

$$H = \begin{pmatrix} h_1(1) & \cdots & h_J(1) \\ \vdots & \ddots & \vdots \\ h_1(M) & \cdots & h_J(M) \end{pmatrix} \in \boldsymbol{R}^{M \times J}.$$

Lemma 8 *If* $\operatorname{rank} H = M$*, then asset markets are effectively complete .*

Note that even if $\operatorname{rank} H = M$, asset markets need not be complete. A sufficient condition for completeness, which involves asset prices, will be given after the proof.

Proof of Lemma 8 Suppose that $(x^1, x^2, \ldots, x^I)$ is a Pareto-efficient contingent-commodity allocation. By Lemma 7, for every i, there exists an $f_i : \{1, 2, \ldots, M\} \to \boldsymbol{R}$ such that

$x^i - e^i = f_i(\chi)$. Write

$$v_i = \begin{pmatrix} f_i(1) \\ \vdots \\ f_i(M) \end{pmatrix} \in \boldsymbol{R}^M,$$

Since $\operatorname{rank} H = M$, there exists a $b_i \in \boldsymbol{R}^J$ such that $v_i = Hb_i$. Then

$$x_t^i - e_t^i = f_i(\chi_t) = Hb^i = \sum_j b^i d_t^j.$$

By Lemma 1, this implies that asset markets are effectively complete. ///

To state the no-retrade theorem, we need the following notation. Let q be an equilibrium asset price process. Since it is adapted to the filtration $(\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_T)$, $q_t^j(s) = q_t^j(s')$ whenever $s_{t'} = s'_{t'}$ for every $t' \leq t$. Thus there exists a $k_j : \bigcup_{\tau=0}^{T} (\{\bar{m}\} \times M^\tau) \to \boldsymbol{R}$ such that $q_t^j(s) = k_j(\chi_0(s), \chi_1(s), \ldots, \chi_t(s))$ for every (s,t). For each t and each $(s_0, s_1, \ldots, s_{t-1})$, define $K(s_0, s_1, \ldots, s_{t-1}) \in \boldsymbol{R}^{M \times J}$ as

$$\begin{pmatrix} h_1(1) + k_1(s_0, s_1, \ldots, s_{t-1}, 1) & \cdots & h_J(1) + k_J(s_0, s_1, \ldots, s_{t-1}, 1) \\ \vdots & \ddots & \vdots \\ h_1(M) + k_1(s_0, s_1, \ldots, s_{t-1}, M) & \cdots & h_J(M) + k_J(s_0, s_1, \ldots, s_{t-1}, M) \end{pmatrix}.$$

While the matrix H represents the dividends of the J assets on the next period, the matrix $K(s_0, s_1, \ldots, s_{t-1})$ represents the total returns to the J assets, inclusive of their prices on the next period. Asset markets are complete if and only if $\operatorname{rank} K(s_0, s_1, \ldots, s_{t-1}) = M$ for every t and $(s_0, s_1, \ldots, s_{t-1})$. In the model of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003), since there are infinitely many periods and the transition probabilities between two states are time-invariant, the asset prices are also time-invariant functions of the M states, and $\operatorname{rank} K(s_0, s_1, \ldots, s_{t-1}) = \operatorname{rank} H = M$ for every t and $(s_0, s_1, \ldots, s_{t-1})$. That is, if $\operatorname{rank} H = M$, then asset markets are complete. In contrast, since asset prices need not be time-invariant functions of the M states in our model, the condition that $\operatorname{rank} H = M$ does not imply that asset markets are complete. It is for this reason that we need to assume that $\operatorname{rank} K(s_0, s_1, \ldots, s_{t-1}) = M$ for every t and $(s_0, s_1, \ldots, s_{t-1})$ in the second part of our no-retrade theorem.

Theorem 7 (No-Retrade Theorem) *Assume that* $\operatorname{rank} H = M$. *If the collection of a feasible contingent-commodity allocation* $(x^1, x^2, \ldots, x^I)$, *a feasible allocation* $(y^1, y^2, \ldots, y^I)$ *of trading plans, and an asset price process* q *is an asset market equilibrium, then there exists a feasible allocation* $(\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^I)$ *of trading plans such that* y^i *is deterministic and constant over time for every* i, *and the collection of* $(x^1, x^2, \ldots, x^I)$, $(\hat{y}^1, \hat{y}^2, \ldots, \hat{y}^I)$, *and* q *is an asset market equilibrium. If, in addition,* $J = M$ *and* $\operatorname{rank} K(s_0, s_1, \ldots, s_{t-1}) = M$ *for every* t *and every* $(s_0, s_1, \ldots, s_{t-1})$, *then* $y^i = \hat{y}^i$ *and hence* y^i *is deterministic and constant over time for every* i.

This theorem states that if asset markets are effectively complete, then any equilibrium contingent-commodity allocation can be attained by letting all consumers trade assets once and for all on period 0, and that if, in addition, markets are complete and the J assets are not redundant, then all consumers do in fact trade assets once and for all on period 0 at equilibrium. Since the equilibrium asset price processes need not be time-invariant, the proof of Theorem 7, which relies on effective completeness, is different from that of Judd, Kubler, and Schmedders (2003) and Kubler and Schmedders (2003), which relies on the stationary dynamic programming technique.

Proof of Theorem 7 Let the collection of a feasible contingent-commodity allocation $(x^1, x^2, \dots, x^I)$, a feasible allocation $(y^1, y^2, \dots, y^I)$ of trading plans, and an asset price process q be an asset market equilibrium. By Lemma 8 and Theorem 2, $(x^1, x^2, \dots, x^I)$ is Pareto-efficient. For each $i \geq 2$, let b^i be as in Lemma 8, let $b^1 = -\sum_{i\geq 2} b^i$, and, for each $i \geq 1$, define $y^i \in X^J$ by letting $y^i_t = b^i$ with probability one for every $t \geq 0$. To show that the collection of $(x^1, x^2, \dots, x^I)$, $(\hat{y}^1, \hat{y}^2, \dots, \hat{y}^I)$, and q is an asset market equilibrium, it suffices to prove that $x^i_0 - e^i_0 \leq -\sum_j q^j_0 \hat{y}^{ji}_0$ for every i. Indeed, there is a $\lambda \in L^1_{++}$ such that (6) holds. Since $d^{y^i} = x^i - e^i = d^{\hat{y}^i}$,

$$\sum_j q^j_0 y^{ji}_0 = \frac{1}{\lambda_t} E\left(\sum_{t=1}^{T} \frac{\lambda_t}{\lambda_0} d^{y^i}_t\right) = E\left(\sum_{t=1}^{T} \frac{\lambda_t}{\lambda_0} d^{\hat{y}^i}_t\right) = \sum_j q^j_0 \hat{y}^{ji}_0.$$

Since $x^i_0 - e^i_0 \leq -\sum_j q^j_0 y^{ji}_0$, $x^i_0 - e^i_0 \leq -\sum_j q^j_0 \hat{y}^{ji}_0$. This completes the proof of the first part.

As for the second part, suppose, in addition, that $J = M$ and $\operatorname{rank} K(s_0, s_1, \dots, s_{t-1}) = M$ for every t and every $(s_0, s_1, \dots, s_{t-1})$. We prove that $y^i = \hat{y}^i$ by a backward induction argument. For each t and $s = (s_0, s_1, \dots, s_T)$, write

$$q(s_0, s_1, \dots, s_t) = \begin{pmatrix} q^1_t(s) \\ \vdots \\ q^J_t(s) \end{pmatrix} \in \boldsymbol{R}^J,$$

$$y_i(s_0, s_1, \dots, s_t) = \begin{pmatrix} y^{1i}_t(s) \\ \vdots \\ y^{Ji}_t(s) \end{pmatrix} \in \boldsymbol{R}^J,$$

$$r_i(s_0, s_1, \dots, s_{t-1}) = \begin{pmatrix} q(s_0, s_1, \dots, s_{t-1}, 1) \cdot y_i(s_0, s_1, \dots, s_{t-1}, 1) \\ \vdots \\ q(s_0, s_1, \dots, s_{t-1}, M) \cdot y_i(s_0, s_1, \dots, s_{t-1}, M) \end{pmatrix} \in \boldsymbol{R}^M.$$

We define $\hat{y}_i(s_0, s_1, \dots, s_t)$ and $\hat{r}_i(s_0, s_1, \dots, s_t)$ analogously for $\hat{y}^i$.

Since $q^j_T = 0$ for every j (because, otherwise, there would be an arbitrage opportunity), (4) with $t = T$ can be rewritten as $v_i = H y^i_{T-1} = H \hat{y}^i_{T-1}$. Since $\operatorname{rank} H = M = J$, this means that $y^i_{T-1} = \hat{y}^i_{T-1}$. As an induction hypothesis, let $t \leq T - 2$ and suppose that $y^i_{t+1} = \hat{y}^i_{t+1}$.

Then (4) can be written as

$$v_i = K(s_0, s_1, \dots, s_{t-1}) y_i(s_0, s_1, \dots, s_{t-1}) - r_i(s_0, s_1, \dots, s_t),$$
$$v_i = K(s_0, s_1, \dots, s_{t-1}) \hat{y}_i(s_0, s_1, \dots, s_{t-1}) - \hat{r}_i(s_0, s_1, \dots, s_t),$$

which is equivalent to

$$K(s_0, s_1, \dots, s_{t-1}) y_i(s_0, s_1, \dots, s_{t-1}) = v_i + r_i(s_0, s_1, \dots, s_t),$$
$$K(s_0, s_1, \dots, s_{t-1}) \hat{y}_i(s_0, s_1, \dots, s_{t-1}) = v_i + \hat{r}_i(s_0, s_1, \dots, s_t).$$

Since $K(s_0, s_1, \dots, s_{t-1})$ is an invertible $M \times M$ matrix and $r_i(s_0, s_1, \dots, s_t) = \hat{r}_i(s_0, s_1, \dots, s_t)$ by the induction hypothesis, $y_i(s_0, s_1, \dots, s_{t-1}) = \hat{y}_i(s_0, s_1, \dots, s_{t-1})$. Thus $y^i_t = \hat{y}^i_t$. ///

Since the theorem holds even when $\operatorname{rank} K(s_0, s_1, \dots, s_{t-1}) < M$ as long as $\operatorname{rank} H = M$, the theorem shows that effective complete asset markets may not be complete.

8 Conclusion

We have proposed a definition of effectively complete asset markets in a model with multiple goods and multiple periods, and established the first welfare theorem in effectively complete asset markets. We have then given four applications of the first welfare theorem, the Pareto efficiency of equilibrium allocations with no aggregate risk, the mutual fund theorem, the sunspot irrelevance theorem, and the no-retrade theorem. The lesson to be learned from this exercise is that the equilibrium allocations may well be Pareto-efficient even in incomplete asset markets, and effective completeness serves as a sufficient (and, in fact, necessary) condition for this to occur.

The usefulness of the concept of effective completeness hinges on to what extent it is applicable. We have presented four distinct classes of economies with effectively complete asset markets. But in all of these classes, asset markets are effectively complete in a very narrow sense, which is that every Pareto-efficient allocation can be attained after the first round of asset trades, without using asset or spot markets in the subsequent rounds. We should, therefore, find other classes of economies with effectively complete asset markets in which multiple rounds of trade are needed to attain Pareto-efficient allocations, although it is, in general, difficult to establish effective completeness involving sequential trades, because the attainability of transfers of contingent commodities through asset trading depends on asset and spot prices in the subsequent rounds.

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