Heterogeneous Beliefs and Mispricing of Derivative Assets

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Abstract

In an exchange economy under uncertainty populated by multiple consumers, we how the heterogeneity in the individual consumers' subjective beliefs affect the representative consumer's utility function. We derive a formula that indicates that the more heterogeneous the individual consumers' beliefs are, the higher probabilities the representative consumer's belief attaches to extreme events that would, in the absence of heterogeneous beliefs, have very low probabilities. We also explore an implication of this formula on derivative asset pricing.

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1 Introduction

In this paper, we study how the heterogeneity in consumers' probabilistic beliefs may affect the pricing of derivative assets. In particular, we show how standard pricing formulas, such as the Black-Scholes option pricing, can be obtained in cases where the consumers have homogeneous beliefs, and introducing heterogeneous beliefs into such cases affect their predictions. In a nutshell, we show that the derivative asset prices predicted in models with heterogeneous beliefs tend to be higher than those predicted in models with homogeneous beliefs.

Whenever we say derivative assets in this paper, we mean that the underlying asset is the (fixed) aggregate consumption of the (exchange) economy. By heterogeneous beliefs, we mean that the distribution functions of the aggregate consumption with respect to consumers' (subjective) beliefs are different. In this respect, our model is the same as that of Huang (2003), but quite different from it in another respect: while Huang (2003) assumed that all consumers believe that the aggregate consumption is log-normally distributed, we cover cases where they believe that the aggregate consumption is distributed according to other types of distributions,

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such as Gamma distributions. Specifically, we assume that the first differential of the log of the density (Radon-Nikodym derivative) of the distribution, according to each consumer's belief, of the aggregate consumption with respect to the true (objective) probability can be multiplicatively separated into two parts, so that it is equal to a function of the realized levels of aggregate consumption, which is common across all consumers, multiplied by a consumer-specific constant. Obscure as it may seem, this assumption is satisfied by surprisingly many families of probability distributions, such as log-normal, Gamma, Chi, Beta, Weibull, Pareto, Poisson, binomial, and negative binomial. Moreover, this assumption is sufficient for the properties called the monotone likelihood ratio condition and the log super-modularity among the individual consumers' beliefs. These properties are well known and often used in the literature on monotone comparative statics.

Our proof method of the result on the mispricing of derivative assets goes as follows. First, we construct the representative consumer along the lines of Wilson (1968) and, under the assumption that all individual consumers have constant and equal relative risk aversion, show that the representative consumer's belief is well defined. Next, we prove that the density of the distribution, according to the representative consumer's belief, of the aggregate consumption with respect to the true (objective) probability has fatter and lower tails than in the (fictitious) case of homogeneous beliefs.[1] It is at this step where the assumption on the heterogeneity of the individual consumers' beliefs is used. Finally, we derive from these fatter tails higher prices for derivative assets, whenever their payoffs are convex functions of the underlying asset (aggregate consumption).

The rest of this paper is organized as follows. In Section 2, we spell out the model and assumptions and give some basic analysis. In Section 3, we establish the results on fatter tails of the density of the representative consumer's belief and the mispricing of derivative assets. In Section 4, we give examples of families of distributions that satisfy our assumption on the heterogeneity of the individual consumers' beliefs. In Section 5, we give a work-out example on how the heterogeneous beliefs induce the density of the representative consumer's belief to have fatter tails. In Section 6, we give a summary of our results and suggest a direction of future research. Some roofs are given in the appendix.

2 Setup

2.1 Uncertainty and consumers

The general setup of our analysis is as follows. Let Z be a non-empty interval in $\boldsymbol{R}$. We represent the uncertainty surrounding the economy by a probability space $(Z, \mathscr{B}(Z), P)$, where $\mathscr{B}(Z)$ is the Borel σ-field of Z. The support of P need not coincide with the entire Z. In particular, it may be a finite subset of Z, and, thus, the case of finitely many states is not really excluded from our analysis although the space Z is an interval. We assume that P is not concentrated

[1] Gollier (2007) gave a general method to relate the representative consumer's belief to the individual consumers' counterparts.

on any single point in Z to exclude the case without uncertainty.

The economy consists of I consumers. Each consumer i has a felicity function $u_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$, which is twice continuously differentiable, and satisfies $u_i'(x_i) > 0 > u_i''(x_i)$ for every $x_i \in \boldsymbol{R}_{++}$ and the Inada condition, that is, $u_i'(x_i) \to 0$ as $x_i \to \infty$, and $u_i'(x_i) \to \infty$ as $x_i \to 0$. Each consumer i also has a subjective probabilistic belief, which is characterized by a probability measure P_i on $(Z, \mathscr{B}(Z))$. His utility function U_i is defined as the expected utility function

$$U_i(c_i) = E^{P_i}(u_i(c_i)) = \int_Z u_i(c_i(z))\,\mathrm{d}P_i(z),$$

where $c_i : Z \to \boldsymbol{R}_{++}$. To be exact, we need to impose some additional restrictions on c_i to make the integral well defined (finite). As such restrictions do not alter our results, we shall not explicitly state or impose them.

We assume that for every i, P_i and P are mutually absolutely continuous. We also assume that there is a version $p_i : Z \to \boldsymbol{R}_{++}$ of the Radon-Nikodym derivative (density) $\mathrm{d}P_i/\,\mathrm{d}P$ that is continuously differentiable on Z.[2] Note that the requirement of continuous differentiability is met whenever P is concentrated on a finite set, or, more generally, whenever there is an at most countable subset Y of Z such that $P(Y) = 1$ and every point of Y is an isolated point of Y. Then

$$U_i(c_i) = \int_Z u_i(c_i(z))p_i(z)\,\mathrm{d}P(z) = E\left(u_i(c_i)p_i\right).$$

The key parameter of the density function is, coined by Wilson (1968),[3] the *dispersion*, $q_i : Z \to \boldsymbol{R}$ defined by $q_i(z) = p_i'(z)/p_i(z)$ for every $z \in Z$. Thus, $q_i(z)$ is equal to the percentage change in the density $p(z)$ when the state variable z is changed by one unit. Moreover,

$$\frac{p_i(z)}{p_i(z^*)} = \exp\left(\int_{z^*}^{z} q_i\left(t\right)\,\mathrm{d}t\right) \tag{1}$$

for all $z \in Z$ and $z^* \in Z$. There is, therefore, a one-to-one relationship between probability density functions and the dispersions.[4]

2.2 Pareto-efficient allocations

To find a Pareto efficient allocation of a given aggregate consumption $c : Z \to \boldsymbol{R}_{++}$ and its supporting (decentralizing) state-price density, it is sufficient to choose positive numbers

[2] Continuous differentiability at extremal points of Z means the existence and continuity of left or right derivatives at such points.

[3] In fact, Wilson (1968) referred to $-p_i'(z)/p_i(z)$ as the dispersion.

[4] Since the subjective probability measure P_i does not uniquely determine the probability density function on sets of P-measure zero, this does not imply that P_i uniquely determines the dispersion, unless the support of the objective probability measure P coincides with the entire Z.

$\lambda_1, \dots, \lambda_I$ and consider the following maximization problem:

$$\begin{aligned} \max_{(c_1,\dots,c_I)} \quad & \sum_i \lambda_i U_i(c_i) \\ \text{subject to} \quad & \sum_i c_i = c. \end{aligned} \tag{2}$$

Since the utility functions U_i are additive with respect to states and the expected utilities are calculated with respect to the common probability measure P, it can be rewritten as

$$\sum_i \lambda_i U_i(c_i) = E\left(\sum_i \lambda_i u_i(c_i)p_i\right) = \int_Z \left(\sum_i \lambda_i u_i(c_i(z))p_i(z)\right) \mathrm{d}P(z).$$

Hence, to solve the original maximization problem (2), it suffices to solve the simplified maximization problem

$$\begin{aligned} \max_{(x_1,\dots,x_I)\in \boldsymbol{R}^I_{++}} \quad & \sum_i \lambda_i u_i(x_i)p_i(z) \\ \text{subject to} \quad & \sum_i x_i = x. \end{aligned} \tag{3}$$

for each pair of a realized aggregate consumption level $x \in \boldsymbol{R}_{++}$ and a state variable $z \in Z$. It can be easily proved that under the stated conditions, there is a unique solution, which we denote by $(f_1(x,z), \dots, f_I(x,z))$. It can also be shown that for each f_i is continuously differentiable in both variables. We can define the value function of this problem $v : \boldsymbol{R}_{++} \times Z \to \boldsymbol{R}$ by

$$v(x,z) = \sum_i \lambda_i u_i\left(f_i(x,z)\right) p_i(z).$$

Then the solution to the original maximization problem is given by $(c^1, \dots, c^I)$, where, for each i, $c^i : Z \to \boldsymbol{R}_{++}$ is defined by $c^i(z) = f_i\left(c(z), z\right)$ for every $z \in Z$. The *representative consumer*'s utility function U is defined by

$$U(c) = \int_Z v(c(z), z)\,\mathrm{d}P(z) = E\left(v(c, \iota)\right),$$

where ι is the identity function on Z. The integrand v_i of each individual consumer's utility function U_i is the product of the felicity function u_i and the density function p_i. This property is called *multiplicative separability.* The integrand v of the representative consumer's utility function U need not have this property. This means, in particular, that the representative consumer's utility function need not be given in the expected utility form.

We define dispersion $q : Z \to \boldsymbol{R}$ for the representative consumer by letting

$$q_i(x,z) = \frac{\dfrac{\partial^2 v(x,z)}{\partial z \partial x}}{\dfrac{\partial v(x,z)}{\partial x}}$$

for every (x,z). This measures the percentage change in the representative consumer's marginal utility when z is increased by one. This definition coincide with the definition of dispersion for the individual consumers if v is multiplicatively separable.

The representative consumer is, of course, not an "actual" consumer, who would trade on financial markets. Rather, he is a theoretical construct, whom we can use to identify asset prices. Specifically, the representative consumer's marginal utility evaluated at the aggregate consumption c, $(\partial u(c,h)/\partial x)$, is a state price density. This means that the price of an asset with dividend $d : Z \to \boldsymbol{R}$, relative to the risk-free bond (which pays off one unit of the commodity whichever state has been realized), is equal to

$$\frac{E\left(\dfrac{\partial v(c,\iota)}{\partial x}d\right)}{E\left(\dfrac{\partial v(c,\iota)}{\partial x}\right)}.$$

Although we analyze the Pareto efficient allocations and their supporting (decentralizing) prices, if the asset markets are complete, then our analysis is applicable to the equilibrium allocations and asset prices. This is because the first welfare theorem holds in complete markets, so that the equilibrium allocations are Pareto efficient and the equilibrium asset prices are given by the corresponding support prices. Since the u_i are concave, the second welfare theorem also holds, so that every Pareto efficient allocation is an equilibrium allocation for some distribution of initial endowments. Hence an analysis of Pareto efficient allocations is also an analysis of equilibrium allocations.

When the solution to the maximization problem (2) is an equilibrium allocation, the individual consumers' wealth shares, evaluated by the equilibrium prices, determines the utility weights λ_i in (2). All the properties we shall explore in the subsequent analysis are valid regardless of the choice of utility weights. Hence, these properties are also valid for the equilibrium allocations regardless of wealth distributions.

2.3 Special case

For the analysis of this paper, we shall concentrate on the special case in which the following assumptions are met

Assumption 1 All individual consumers share the same constant relative risk aversion $\gamma > 0$.

Thus, we can assume without loss of generality that they share the same felicity function, which is denoted by $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ satisfying $u'(x) = x^{-\gamma}$ for every $x \in \boldsymbol{R}_{++}$. This assumption is imposed to ensure that any deviation from the standard results on derivative asset pricing is due to the heterogeneity in subjective probabilistic beliefs, not in risk attitudes.

Under Assumption 1, the first-order condition for the solution to the maximization problem (2) is that

$$\lambda_i(f_i(x,z))^{-\gamma}p_i(z) = \frac{\partial v}{\partial x}(x,z)$$

for every i and $(x, z) \in \boldsymbol{R}_{++} \times Z$. Rearranging this, summing them over i, and the rearranging the resulting expression, we obtain

$$\frac{\partial v}{\partial x}(x, z) = \left(\sum_i (\lambda_i p_i(z))^{1/\gamma} \right)^{\gamma} x^{-\gamma}$$

for every $z \in Z$. We can, thus, let the representative consumer's utility function be the same as the individual consumers' u (having constant relative risk aversion equal to γ) and his belief be given by the density function $p : Z \to \boldsymbol{R}_{++}$ defined by

$$p(z) = \frac{\left(\sum_i (\lambda_i p_i(z))^{1/\gamma} \right)^{\gamma}}{E\left(\left(\sum_i (\lambda_i p_i(c))^{1/\gamma} \right)^{\gamma} \right)}$$

for every $z \in Z$. That is,

$$v(x, z) = p(z)u(x) = \frac{\left(\sum_i (\lambda_i p_i(z))^{1/\gamma} \right)^{\gamma} u(x)}{E\left(\left(\sum_i (\lambda_i p_i(c))^{1/\gamma} \right)^{\gamma} \right)}.$$

A straightforward calculation shows that his dispersion, $q = p'/p$, is given by

$$q(z) = \frac{\sum_i q_i(z) (\lambda_i p_i(z))^{1/\gamma}}{\sum_i (\lambda_i p_i(z))^{1/\gamma}}$$

for every $z \in Z$.

We impose the following assumption on the way in which the subjective beliefs are heterogeneous.

Assumption 2 There exists a Borel-measurable function $g : Z \to \boldsymbol{R}_{++}$ such that for every i, there exists an $\alpha_i \in \boldsymbol{R}$ such that $q_i(z) = \alpha_i g(z)$ for every $z \in Z$.

This assumption requires that all individual consumers' dispersions be heterogeneous only in terms of the scaling factors of some common function defined on Z. Since $q_i = p_i'/p_i$, this condition is equivalent to the following one: There exists a continuous function $g : Z \to \boldsymbol{R}_{++}$ such that for every i and every $z^\circ \in Z$, there exists an $(\alpha_i, \beta_i) \in \boldsymbol{R}^2$ such that

$$p_i(z) = \exp\left(\alpha_i \int_{z^\circ}^{z} g(t)\, \mathrm{d}t + \beta_i \right)$$

for every $z \in Z$. Here β_i is the scaling factor, determined uniquely by α_i and z°, that leads to $\int_Z p_i(z)\, \mathrm{d}P(z) = 1$.

Define $q(\cdot|\cdot) : Z \times \boldsymbol{R} \to \boldsymbol{R}$ by letting $q(z|\alpha) = \alpha g(z)$. Define $p(\cdot|\cdot) : Z \times \boldsymbol{R} \to \boldsymbol{R}_{++}$ by letting $p(\cdot|\alpha)$ be the probability density function corresponding to $q(\cdot|\alpha)$ (that is, $\partial \ln p(z|\alpha)/\partial z = q(z|\alpha)$). Let $P(\cdot|\alpha) : \mathscr{B}(Z) \to [0, 1]$ be the probability measure corresponding to $p(\cdot|\alpha)$ (that is, $\mathrm{d}P(\cdot|\alpha)/\, \mathrm{d}P = p(\cdot|\alpha)$). Then the family $(P(\cdot|\alpha))_{\alpha \in \boldsymbol{R}}$ of probability measures satisfies the monotone likelihood ratio condition, or, equivalently, $p(\cdot|\cdot) : Z \times \boldsymbol{R} \to \boldsymbol{R}_{++}$ is log-supermodular,[5] because

$$\frac{\partial^2}{\partial\alpha\partial z} \ln p(z|\alpha) = \frac{\partial}{\partial\alpha} q(z|\alpha) = g(z) > 0.$$

According to this notation, $q_i(z) = q(z|\alpha_i)$, $p_i(z) = p(z|\alpha_i)$, and $P_i = P(\cdot|\alpha_i)$.

The crucial property embedded in Assumption 2 is the *multiplicative separability* between α and z. Indeed, if there exist a strictly increasing or strictly decreasing function $k : \boldsymbol{R} \to \boldsymbol{R}$ such $q(z|\alpha) = k(\alpha)g(z)$, then we could simply refer to $k(\alpha)$ or $-k(\alpha)$ as α to satisfy Assumption 2.

We now explore some properties of the representative consumer's belief and the state-price density. First, define $h_i : Z \to \boldsymbol{R}_{++}$ and $m : Z \to \boldsymbol{R}_{++}$ by letting

$$\begin{aligned} h_i(z) &= (\lambda_i p_i(z))^{1/\gamma}, \\ m(z) &= \frac{\sum_i \alpha_i h_i(z)}{\sum_i h_i(z)} \end{aligned}$$

for every $z \in Z$. Then $q(z) = g(z)m(z)$ for every $z \in Z$. This, in particular, implies that if all individual consumers have the same scaling factor, then the representative consumer also has the same scaling factor. Moreover, m is non-decreasing, because, for every $z \in Z$,

$$m'(z) = \frac{1}{\gamma}\left(\frac{\sum_i \alpha_i^2 h_i(z)}{\sum_i h_i(z)} - \left(\frac{\sum_i \alpha_i h_i(z)}{\sum_i h_i(z)}\right)^2\right) = \frac{1}{\gamma}\frac{\sum_i (\alpha_i - m(z))^2 h_i(z)}{\sum_i h_i(z)}.$$

The last term is strictly positive, and m is strictly increasing, under the following assumption.

Assumption 3 Under Assumption 2, there are i and j such that $\alpha_i \neq \alpha_j$.

We end this subsection by introducing the concept of the *normalized* state price density. Let $c : Z \to \boldsymbol{R}_{++}$ be the aggregate consumption. Using the representative consumer's utility function u and probability density function p, we define $\pi : Z \to \boldsymbol{R}_{++}$ by letting

$$\pi(z) = \frac{p(z)u'(c(z))}{E(pu'(c))}. \tag{4}$$

Then $E(\pi) = 1$. Thus the the risk-free asset is the numeraire for π, from which the term "normalized state-price deflator" is derived. For $p(\cdot|\alpha)$, we define the corresponding normalized state price density $\pi(\cdot|\alpha) : Z \to \boldsymbol{R}_{++}$ by letting

$$\pi(z|\alpha) = \frac{p(z|\alpha)u'(c(z))}{E(p(\cdot|\alpha)(u'(c))}. \tag{5}$$

3 Main results

In this section, we show how the heterogeneity in individual consumers' subjective beliefs lead to mispricing of derivatives, such as options.

[5] Another equivalent condition is that $\ln p(\cdot|\cdot)$ satisfies the increasing difference condition.

To state our main theorem, we need to fix our terminology. Let $c : Z \to \boldsymbol{R}_{++}$ be the aggregate consumption. By a derivative asset, we mean a derivative asset whose underlying asset is the aggregate consumption. Thus its payoff is determined by a function $\varphi : \boldsymbol{R}_{++} \to \boldsymbol{R}$ so that $\varphi(c(z))$ is equal to the payoff of the derivative asset in state z.

By mispricing of derivative asset prices, we mean, roughly, that if all individual consumers are erroneously assumed to share the same scaling factor of dispersion, then the derivative asset whose payoff is a nonlinear function of the aggregate consumption must necessarily be mispriced. To formalize this idea, recall first that if all individual consumers have the same scaling factor α of dispersion, then the representative consumer also has the same scaling factor α of dispersion. Thus, for the purpose of asset pricing, assuming that all individual consumers have the same scaling factor α of dispersion amounts to assuming that the representative consumer has the scaling factor α of dispersion. We shall therefore compare the representative consumer's "true" probability density and the "true" state-price density, taking heterogeneous beliefs into consideration, with their "fictitious" counterparts when the representative consumer is assumed to have a constant scaling factor of dispersion.

The following assumption is used in the main results of this paper.

Assumption 4 The aggregate consumption $c : Z \to \boldsymbol{R}_{++}$ is a strictly increasing function.

This assumption implies that the orderings of the aggregate consumption and Z coincide. That is, the aggregate consumption is low at the lower tail of the interval Z and it is high at the higher tail of interval Z. We can now state the first main result of this paper.

Theorem 1 *Under Assumptions 1, 2, 3, and 4, let $\alpha \in \boldsymbol{R}$.*

1. *If $E(p(\cdot|\alpha)c) = E(pc)$, then there are $z_1 \in Z$ and $z_2 \in Z$ such that $z_1 < z_2$, $p(z) < p(z|\alpha)$ whenever $z_1 < z < z_2$, and $p(z) > p(z|\alpha)$ whenever $z < z_1$ or $z > z_2$.*

2. *If $E(\pi(\cdot|\alpha)c) = E(\pi c)$, then there are $z_1 \in Z$ and $z_2 \in Z$ such that $z_1 < z_2$, $\pi(z) < \pi(z|\alpha)$ whenever $z_1 < z < z_2$, and $\pi(z) > \pi(z|\alpha)$ whenever $z < z_1$ or $z > z_2$.*

The equality $E(p(\cdot|\alpha)c) = E(pc)$ means that the value of the scaling factor α in the dispersion is chosen so that the mean of the aggregate consumption is correctly calculated even when the heterogeneity in subjective beliefs is ignored. Part 1 of the theorem can thus be interpreted as saying that the heterogeneity in subjective beliefs puts more weights on the upper and lower tails of the probability density than in the absence of heterogeneity. Since $(p(\cdot|\cdot))$ is supermodular, for any two distinct α and α' in $\boldsymbol{R}$, the two probability density functions $p(\cdot|\alpha)$ and $p(\cdot|\alpha')$ cross each other only once. Part 1, therefore, implies that the representative consumer's probability density function p does not belong to the family $(p(\cdot|\alpha))_{\alpha\in\boldsymbol{R}}$ of probability density functions to which all individual consumers' probability density functions belong. In this sense, the family $(p(\cdot|\alpha))_{\alpha\in\boldsymbol{R}}$ of probability density functions is not closed under aggregation.

The equality $E(\pi(\cdot|\alpha)c) = E(\pi c)$ means that the value of the scaling factor α in the dispersion is chosen so that the prices of the aggregate consumption is correctly calculated even when the heterogeneity in subjective beliefs is ignored. Part 2 of the theorem can thus be interpreted as saying that the heterogeneity in subjective beliefs puts more weights on the upper and lower tails of the state-price density than in the absence of heterogeneity. It will be applied to the next theorem, which identifies the bias in the prediction of derivative asset prices when the belief heterogeneity is ignored.

The proof of Theorem 1 is given in the appendix. We now move on to the second main theorem of this paper.

Theorem 2 *Under Assumptions 1, 2, 3, and 4, let* $\alpha \in \boldsymbol{R}$ *and suppose that* $E(\pi(\cdot|\alpha)c) = E(\pi c)$.

1. *If the derivative asset* $\varphi : \boldsymbol{R}_{++} \to \boldsymbol{R}$ *is convex on* $\boldsymbol{R}_{++}$ *but not linear on the support of* P,[6] *then* $E(\pi(\cdot|\alpha)\varphi(c)) < E(\pi\varphi(c))$.
2. *If the derivative asset* $\varphi : \boldsymbol{R}_{++} \to \boldsymbol{R}$ *is concave on* $\boldsymbol{R}_{++}$ *but not linear on the support of* P, *then* $E(\pi(\cdot|\alpha)\varphi(c)) > E(\pi\varphi(c))$.

This theorem formalizes the idea that heterogeneous beliefs leads to mispricing of derivative assets. Part 1 claims that if the payoff of the derivative asset is a convex function of its underlying asset, the aggregate consumption in our case, then ignoring the heterogeneity in beliefs and using the state-price density $\pi(\cdot|\alpha)$ predicts a lower price than the true price. Since the payoffs of both call and put options are convex functions of that of the underlying asset, this means that they are underpriced if the heterogeneity in beliefs are ignored. Part 2 is in the same vein: if the payoff of the derivative asset is a concave function of that of its underlying asset, then ignoring the heterogeneity in beliefs and using the state-price density $\pi(\cdot|\alpha)$ predicts a lower price than the true price. The following proof is much due to Franke, Stapleton, and Subrahmanyam (1999, Lemma 1).

Proof of Theorem 2 Let $\varphi : \boldsymbol{R}_{++} \to \boldsymbol{R}$ be convex on $\boldsymbol{R}_{++}$ and not linear on the support of P. By Assumption 4 and the choice of α, there exist $z_1 \in Z$ and $z_2 \in Z$ having the properties in Part 2 of Theorem 1. Define $\psi : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by

$$\psi(x) = \frac{\varphi(c(z_2)) - \varphi(c(z_1))}{c(z_2) - c(z_1)}(x - c(z_1)) + \varphi(c(z_1)).$$

Then $\psi(x) \geq \varphi(x)$ whenever $c(z_1) \leq x \leq c(z_2)$, and $\psi(x) \leq \varphi(x)$ whenever $x \leq c(z_1)$ or $c(z_2) \leq x$; and a strict inequality holds on some set of z's of positive measure. Thus,

$$(\pi(z) - \pi(z|\alpha))\,(\varphi(c(z)) - \psi(c(z))) \geq 0$$

for every $z \in Z$, and a strict inequality holds on some some set of z's of positive measure. Hence

$$E\left((\pi - \pi(\cdot|\alpha))\,(\varphi(c) - \psi(c))\right) > 0.$$

[6] That is, there is no linear function $\psi : \boldsymbol{R} \to \boldsymbol{R}$ that coincides with φ on the support of P.

But since

$$\psi(c)=\frac{\varphi(c(z_2))-\varphi(c(z_1))}{c(z_2)-c(z_1)}c+\left(\varphi(c(z_1))-\frac{\varphi(c(z_2))-\varphi(c(z_1))}{c(z_2)-c(z_1)}c(z_1)\right),$$

$\psi(c)$ is a scalar multiple of c added by a constant. Hence, by the choice of α, $E\left((\pi-\pi(\cdot|\alpha))\,\psi(c)\right)=0$. Therefore, $E\left((\pi-\pi(\cdot|\alpha))\,\varphi(c)\right)>0$, that is, $E\left(\pi(\cdot|\alpha)\varphi(c)\right)<E\left(\pi\varphi(c)\right)$. This completes the proof of Part 1.

Part 2 can be proven analogously. ///

4 Distributions

Assumption 2 requires that the individual consumers' dispersions be scalar multiples of one another. In this section, we give several classes of parametric examples of probability distributions for which Assumption 2 is satisfied.

4.1 Normal distributions

Assume that $Z=\boldsymbol{R}$ and P is the normal distribution with mean μ and variance σ^2. Assume that for every i, P_i is the normal distribution with mean μ_i and variance σ_i^2. Then, for every i and $z\in Z$,

$$\begin{aligned}p_i(z) &= \frac{\dfrac{1}{(2\pi)^{1/2}\,\sigma_i}\exp\left(-\dfrac{(z-\mu_i)^2}{2\sigma_i^2}\right)}{\dfrac{1}{(2\pi)^{1/2}\,\sigma}\exp\left(-\dfrac{(z-\mu)^2}{2\sigma^2}\right)}\\ &= \frac{\sigma}{\sigma_i}\exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_i^2}-\frac{1}{\sigma^2}\right)z^2+\left(\frac{\mu_i}{\sigma_i^2}-\frac{\mu}{\sigma^2}\right)z-\frac{1}{2}\left(\frac{\mu_i^2}{\sigma_i^2}-\frac{\mu^2}{\sigma^2}\right)\right).\end{aligned}$$

Hence,

$$q_i(z)=-\left(\frac{1}{\sigma_i^2}-\frac{1}{\sigma^2}\right)z+\left(\frac{\mu_i}{\sigma_i^2}-\frac{\mu}{\sigma^2}\right).$$

Thus, Assumption 2 is met if $\sigma_i=\sigma$ for every i (in which case $g(z)=1$ and $\alpha_i=(\mu_i-\mu)/\sigma^2$).

4.2 Log-normal distributions

Assume that $Z=\boldsymbol{R}_{++}$ and P is the log-normal distribution with parameters μ and σ. Assume that for every i, P_i is the log-normal distribution with parameters μ_i and σ_i. Then, for every i

and $z \in Z$,

$$p_i(z) = \frac{\dfrac{1}{(2\pi)^{1/2}\,\sigma_i z} \exp\left(-\dfrac{(\ln z - \mu_i)^2}{2\sigma_i^2}\right)}{\dfrac{1}{(2\pi)^{1/2}\,\sigma z} \exp\left(-\dfrac{(\ln z - \mu)^2}{2\sigma^2}\right)}$$
$$= \frac{\sigma}{\sigma_i} \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_i^2} - \frac{1}{\sigma^2}\right)(\ln z)^2 + \left(\frac{\mu_i}{\sigma_i^2} - \frac{\mu}{\sigma^2}\right)\ln z - \frac{1}{2}\left(\frac{\mu_i^2}{\sigma_i^2} - \frac{\mu^2}{\sigma^2}\right)\right).$$

Hence,

$$q_i(z) = -\left(\frac{1}{\sigma_i^2} - \frac{1}{\sigma^2}\right)\frac{\ln z}{z} + \left(\frac{\mu_i}{\sigma_i^2} - \frac{\mu}{\sigma^2}\right)\frac{1}{z}.$$

Thus, Assumption 2 is met if $\sigma_i = \sigma$ for every i (in which case $g(z) = 1/z$ and $\alpha_i = (\mu_i - \mu)/\sigma^2$).

4.3 Gamma distributions

Assume that $Z = \boldsymbol{R}_{++}$ and P is the gamma distribution with parameters κ and θ. Assume that for every i, P_i is the gamma distribution with parameters κ_i and θ_i. Then, for every i and $z \in Z$,

$$p_i(z) = \frac{\dfrac{\theta_i^{\kappa_i}}{\Gamma(\kappa_i)} z_i^{\kappa_i - 1} \exp(-\theta_i z)}{\dfrac{\theta^{\kappa}}{\Gamma(\kappa)} z^{\kappa - 1} \exp(-\theta z)} = \frac{\theta_i^{\kappa_i}}{\Gamma(\kappa_i)} \frac{\Gamma(\kappa)}{\theta^{\kappa}} z^{\kappa_i - \kappa} \exp(-(\theta_i - \theta) z),$$

where Γ is the gamma function:

$$\Gamma(\kappa) = \int_0^{\infty} z^{\kappa - 1} \exp(-z)\, \mathrm{d}z.$$

Hence,

$$q_i(z) = (\kappa_i - \kappa)\frac{1}{z} - (\theta_i - \theta).$$

Thus, Assumption 2 is met if $\kappa_i = \kappa$ for every i (in which case $g(z) = 1$ and $\alpha_i = -(\theta_i - \theta)$) or if $\theta_i = \theta$ for every i (in which case $g(z) = 1/z$ and $\alpha_i = \kappa_i - \kappa$).

4.4 Inverse gamma distributions

Assume that $Z = \boldsymbol{R}_{++}$ and P is the inverse gamma distribution with parameters κ and θ. Assume that for every i, P_i is the inverse gamma distribution with parameters κ_i and θ_i. Then, for every i and $z \in Z$,

$$p_i(z) = \frac{\dfrac{\theta_i^{\kappa_i}}{\Gamma(\kappa_i)} \dfrac{1}{z^{\kappa_i + 1}} \exp\left(-\dfrac{\theta_i}{z}\right)}{\dfrac{\theta^{\kappa}}{\Gamma(\kappa)} \dfrac{1}{z^{\kappa + 1}} \exp\left(-\dfrac{\theta}{z}\right)} = \frac{\theta_i^{\kappa_i}}{\Gamma(\kappa_i)} \frac{\Gamma(\kappa)}{\theta^{\kappa}} z^{-(\kappa_i - \kappa)} \exp\left(-(\theta_i - \theta)\frac{1}{z}\right).$$

Hence,

$$q_i(z) = -(\kappa_i - \kappa)\frac{1}{z} + (\theta_i - \theta)\frac{1}{z^2}.$$

Thus, Assumption 2 is met if $\kappa_i = \kappa$ for every i (in which case $g(z) = 1/z^2$ and $\alpha_i = \theta_i - \theta$) or if $\theta_i = \theta$ for every i (in which case $g(z) = 1/z$ and $\alpha_i = -(\kappa_i - \kappa)$).

4.5 Chi distributions

Assume that $Z = \boldsymbol{R}_{++}$ and P is the chi distribution with the degree κ of freedom. Assume that for every i, P_i is the gamma distribution with the degree κ_i of freedom. Then, for every i and $z \in Z$,

$$p_i(z) = \frac{\dfrac{2^{1-\kappa_i/2}}{\Gamma\left(\dfrac{\kappa_i}{2}\right)} z^{\kappa_i - 1} \exp\left(-\dfrac{z^2}{2}\right)}{\dfrac{2^{1-\kappa/2}}{\Gamma\left(\dfrac{\kappa}{2}\right)} z^{\kappa - 1} \exp\left(-\dfrac{z^2}{2}\right)} = 2^{-(\kappa_i - \kappa)/2} \frac{\Gamma\left(\dfrac{\kappa}{2}\right)}{\Gamma\left(\dfrac{\kappa_i}{2}\right)} z^{\kappa_i - \kappa}.$$

Hence,

$$q_i(z) = (\kappa_i - \kappa)\frac{1}{z}.$$

Thus, Assumption 2 is met, with $g(z) = 1/z$ and $\alpha_i = \kappa_i - \kappa$.

4.6 Beta distributions

Assume that $Z = (0, 1)$ and P is the beta distribution with parameters η and κ. Assume that for every i, P_i is the Pareto distribution with parameters η_i and κ_i. Then, for every i and $z \in Z$,

$$p_i(z) = \frac{\dfrac{z^{\eta_i - 1} z^{\kappa_i - 1}}{\mathrm{B}(\eta_i, \kappa_i)}}{\dfrac{z^{\eta - 1} z^{\kappa - 1}}{\mathrm{B}(\eta, \kappa)}},$$

where B is the beta function:

$$\mathrm{B}(\eta, \kappa) = \int_0^1 z^{\eta - 1} z^{\kappa - 1} \, \mathrm{d}z.$$

Hence,

$$q_i(z) = (\eta_i - \eta)\frac{1}{z} - (\kappa_i - \kappa)\frac{1}{1 - z}.$$

Thus, Assumption 2 is met if $\eta_i = \eta = 0$ for every i (in which case $g(z) = 1/(1 - z)$ and $\alpha_i = \kappa_i - \kappa$) or if $\kappa_i = \kappa$ for every i (in which case $g(z) = 1/z$ and $\alpha_i = -(\eta_i - \eta)$).

4.7 Weibull distributions

Assume that $Z = \boldsymbol{R}_{++}$ and P is the Weibull distribution with parameters κ and θ. Assume that for every i, P_i is the Weibull distribution with parameters κ_i and θ_i. Then, for every i and

$z \in Z$,

$$p_i(z) = \frac{\dfrac{\kappa_i}{\theta_i}\left(\dfrac{z}{\theta_i}\right)^{\kappa_i - 1} \exp\left(-\left(\dfrac{z}{\theta_i}\right)^{\kappa_i}\right)}{\dfrac{\kappa}{\theta}\left(\dfrac{z}{\theta}\right)^{\kappa - 1} \exp\left(-\left(\dfrac{z}{\theta}\right)^{\kappa}\right)} = \frac{\kappa_i}{\theta_i^{\kappa_i}}\frac{\theta^\kappa}{\kappa} z^{\kappa_i - \kappa} \exp\left(-\left(\left(\frac{z}{\theta_i}\right)^{\kappa_i} - \left(\frac{z}{\theta}\right)^{\kappa}\right)\right)$$

Hence,

$$q_i(z) = (\kappa_i - \kappa)\frac{1}{z} - \left(\frac{\kappa_i}{\theta_i^{\kappa_i}} z^{\kappa_i - 1} - \frac{\kappa}{\theta^\kappa} z^{\kappa - 1}\right).$$

Thus, if $\kappa_i = \kappa$ for every i, then

$$q_i(z) = -\left(\frac{\kappa}{\theta_i^\kappa} - \frac{\kappa}{\theta^\kappa}\right) z^{\kappa - 1},$$

and Assumption 2 is met, with $g(z) = z^{\kappa-1}$ and $\alpha_i = -\left(\kappa/\theta_i^\kappa - \kappa/\theta^\kappa\right)$.

4.8 Pareto distributions

Let $\hat{z} \in \boldsymbol{R}_{++}$. Assume that $Z = (\hat{z}, \infty)$ and P is the Pareto distribution with parameter κ. Assume that for every i, P_i is the Pareto distribution (on $(\hat{z}, \infty)$) with parameter κ_i. Then, for every i and $z \in Z$,

$$p_i(z) = \frac{\dfrac{\kappa_i \hat{z}^{\kappa_i}}{z^{\kappa_i + 1}}}{\dfrac{\kappa \hat{z}^{\kappa}}{z^{\kappa + 1}}} = \frac{\kappa_i}{\kappa}\hat{z}^{\kappa_i - \kappa} z^{-(\kappa_i - \kappa)}.$$

Hence,

$$q_i(z) = -(\kappa_i - \kappa)\frac{1}{z}$$

and Assumption 2 is met, with $g(z) = 1/z$ and $\alpha_i = -(\kappa_i - \kappa)$.

4.9 Poisson distributions

Assume that $Z = \boldsymbol{R}_+$ and P is the Poisson distribution with parameter θ. Assume that for every i, P_i is the Poisson distribution with parameters θ_i. Then, for every i and $z \in \{0, 1, \dots\}$,

$$p_i(z) = \frac{\dfrac{\exp(-\theta_i)\theta_i^z}{z!}}{\dfrac{\exp(-\theta)\theta^z}{z!}} = \left(\frac{\theta_i}{\theta}\right)^z \exp(-(\theta_i - \theta)).$$

Although the probability mass function of any Poisson distribution can be defined only at non-negative integers, the ratio of the probability mass functions of two Poisson distributions can be defined at all real numbers. Hence

$$q_i(z) = \ln\frac{\theta_i}{\theta}$$

Thus, Assumption 2 is met, with $g(z) = 1$ and $\alpha_i = \ln(\theta_i/\theta)$.

4.10 Binomial distributions

Assume that $Z = \boldsymbol{R}_+$ and P is the binomial distribution with n trials and probability θ of success. Assume that for every i, P_i is the binomial distribution with n trials and probability θ_i of success. Then, for every i and $z \in \{0, 1, \dots, n\}$,

$$p_i(z) = \frac{\begin{pmatrix} n \\ z \end{pmatrix} \theta_i^z (1-\theta_i)^{n-z}}{\begin{pmatrix} n \\ z \end{pmatrix} \theta^z (1-\theta)^{n-z}} = \left(\frac{\theta_i(1-\theta)}{\theta(1-\theta_i)}\right)^z \left(\frac{1-\theta_i}{1-\theta}\right)^n$$

Hence,

$$q_i(z) = \ln \frac{\theta_i(1-\theta)}{\theta(1-\theta_i)} = \ln \theta_i(1-\theta) - \ln \theta(1-\theta_i).$$

Thus, Assumption 2 is met, with $g(z) = 1$ and $\alpha_i = \ln \theta_i(1-\theta) - \ln \theta(1-\theta_i)$.

4.11 Negative binomial distributions

Assume that $Z = \boldsymbol{R}_+$ and P is the negative binomial distribution with the n successes required and probability θ of success. Assume that for every i, P_i is the negative binomial distribution with n_i successes required and probability probability θ_i of success. Then, for every i and $z \in \{0, 1, \dots, \}$,

$$p_i(z) = \frac{\begin{pmatrix} z + n_i - 1 \\ n_i - 1 \end{pmatrix} \theta_i^{n_i} (1-\theta_i)^z}{\begin{pmatrix} z + n - 1 \\ n - 1 \end{pmatrix} \theta^n (1-\theta)^z}$$

Hence, if $n_i = n$ for every i, then

$$p_i(z) = \left(\frac{\theta_i}{\theta}\right)^n \left(\frac{1-\theta_i}{1-\theta}\right)^z$$

and

$$q_i(z) = \ln \frac{1-\theta_i}{1-\theta} = \ln(1-\theta_i) - \ln(1-\theta).$$

Thus, if $n_i = n$ for every i, then Assumption 2 is met, with $g(z) = 1$ and $\alpha_i = \ln(1-\theta_i) - \ln(1-\theta)$.

5 Example

In this section, we give a work-out example on how the heterogeneity in individual consumers' beliefs induces fatter tails of the representative consumer's belief. This may be used to explain how the "true" option price deviates from the price predicted by the Black-Scholes formula.

First, we let $Z = \boldsymbol{R}$ and P be a normal distribution with mean 0 and variance t, where $t \in \boldsymbol{R}_{++}$. Here t is to be interpreted as the time to maturity of the European call option. Second, we extend our setting to the case where there are infinitely many consumers by letting

$(I, \mathscr{I}, \nu)$ be the probability space of (the names of) consumers in the economy. As before, all of them have constant and equal constant relative risk aversion γ. For each i, consumer i's probabilistic belief P_i is specified by its Radon-Nikodym derivative $p_i = \mathrm{d}P_i / \mathrm{d}P$ with respect to the objective probability P, which is defined by

$$p_i(z) = \exp\left(\mu(i)z - \frac{(\mu(i))^2 t}{2}\right).$$

This means that P_i is the normal distribution with mean $\mu(i)t$ and variance t. Indeed, denote by Λ the Lebesgue measure on $\boldsymbol{R}$, then

$$\frac{\mathrm{d}P}{\mathrm{d}\Lambda}(z) = (2\pi t)^{-1/2} \exp\left(-\frac{z^2}{2t}\right),$$

and hence

$$\begin{aligned}\frac{\mathrm{d}P_i}{\mathrm{d}\Lambda}(z) &= \frac{\mathrm{d}P_i}{\mathrm{d}P}(z)\frac{\mathrm{d}P}{\mathrm{d}\Lambda}(z) \\ &= \exp\left(\mu(i)z - \frac{(\mu(i))^2 t}{2}\right)(2\pi t)^{-1/2} \exp\left(-\frac{z^2}{2t}\right) \\ &= (2\pi t)^{-1/2} \exp\left(-\frac{(z-\mu(i)t)^2}{2t}\right).\end{aligned}$$

To accommodate a continuum of consumers, the welfare maximization problem (2) is modified so that

$$\begin{aligned}\max_{(c_i)_{i\in I}} \quad & \int_I \lambda(i) U_i(c_i)\, \mathrm{d}\nu(i) \\ \text{subject to} \quad & \int_I c_i \,\mathrm{d}\nu(i) = c.\end{aligned} \tag{6}$$

The welfare maximization problem (3) is modified so that

$$\begin{aligned}\max_{(x_i)_{i\in I}\in \boldsymbol{R}^I_{++}} \quad & \int_I \lambda(i) u_i(x_i) p_i(z)\, \mathrm{d}\nu(i) \\ \text{subject to} \quad & \int_I x_i \,\mathrm{d}\nu(i) = x.\end{aligned} \tag{7}$$

Then the representative consumer's belief P_0 is given by the Radon-Nikodym derivative[7]

$$
\begin{aligned}
p(z) &= \frac{\left(\displaystyle\int_I (\lambda(i)p_i(z))^{1/\gamma}\,\mathrm{d}\nu(i)\right)^{\gamma}}{E\left(\left(\displaystyle\int_I (\lambda(i)p_i(c))^{1/\gamma}\,\mathrm{d}\nu(i)\right)^{\gamma}\right)} \\
&= \frac{\left(\displaystyle\int_I \exp\left(\frac{\mu(i)z}{\gamma}-\frac{(\mu(i))^2 t}{2\gamma}\right)(\lambda(i))^{1/\gamma}\,\mathrm{d}\nu(i)\right)^{\gamma}}{E\left(\left(\displaystyle\int_I \exp\left(\frac{\mu(i)c}{\gamma}-\frac{(\mu(i))^2 t}{2\gamma}\right)(\lambda(i))^{1/\gamma}\,\mathrm{d}\nu(i)\right)^{\gamma}\right)}.
\end{aligned} \tag{8}
$$

By multiplying a positive constant if necessary, we can assume that $\int_I (\lambda(i))^{1/\gamma}\,\mathrm{d}\nu(i) = 1$. It can be shown that if all the $\mu(i)$ were equal, then the solution $(f_i(x,z))_{i\in I}$ to (3) would be given by $f_i(x,z) = (\lambda(i))^{1/\gamma}x$ for every (x,z). Thus, if there were no heterogeneity, then $(\lambda(i))^{1/\gamma}$ would be equal to the consumption share of consumer i in the aggregate consumption. Since this share does not depend on z, it would also be the wealth share of consumer i. When, in fact, the beliefs are heterogeneous, $f_i(x,z)$ depends on z and $(\lambda(i))^{1/\gamma}$ does not coincide the wealth share of consumer i. Yet, Lemma 4.1 of Jouini and Napp (2007) shows that it approximates the wealth share. Define a probability measure ν^* on $\boldsymbol{R}$ by

$$
\nu^*(B) = \int_{\mu^{-1}(B)} (\lambda(i))^{1/\gamma}\,\mathrm{d}\nu(i)
$$

for every $B \in \mathscr{B}(\boldsymbol{R})$. Then ν^* approximates the distribution of the $\mu(i)$ in terms of the wealth share in the economy.

We assume that ν^* is the normal distribution with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$. Then, by the change-of-variable formula,

$$
\begin{aligned}
&\int_I \exp\left(\frac{\mu(i)z}{\gamma}-\frac{(\mu(i))^2 t}{2\gamma}\right)(\lambda(i))^{1/\gamma}\,\mathrm{d}\nu(i) \\
&= \int_{\boldsymbol{R}} \exp\left(\frac{qz}{\gamma}-\frac{q^2 t}{2\gamma}\right)\mathrm{d}\nu^*(q) \\
&= \int_{-\infty}^{\infty} \exp\left(\frac{qz}{\gamma}-\frac{q^2 t}{2\gamma}\right)(2\pi\hat{\sigma}^2)^{-1/2}\exp\left(-\frac{(q-\hat{\mu})^2}{2\hat{\sigma}^2}\right)\mathrm{d}q.
\end{aligned}
$$

Here

$$
\begin{aligned}
&\exp\left(\frac{qz}{\gamma}-\frac{q^2 t}{2\gamma}\right)\exp\left(-\frac{(q-\hat{\mu})^2}{2\hat{\sigma}^2}\right) \\
&= \exp\left(-\frac{1}{2}\left(\frac{t}{\gamma}+\frac{1}{\hat{\sigma}^2}\right)\left(q-\frac{\gamma\hat{\mu}+\hat{\sigma}^2 z}{\gamma+\hat{\sigma}^2 t}\right)^2\right)\exp\left(\frac{1}{2}\frac{(\gamma\hat{\mu}+\hat{\sigma}^2 z)^2}{\gamma\hat{\sigma}^2(\gamma+\hat{\sigma}^2 t)}-\frac{1}{2}\frac{\hat{\mu}^2}{\hat{\sigma}^2}\right).
\end{aligned}
$$

[7] To be precise, we need to assume that the functions $i \mapsto \mu(i)$ and $i \mapsto \lambda(i)$ are measurable and satisfy some integrability conditions.

Thus,

$$\int_I \exp\left(\frac{\mu(i)z}{\gamma} - \frac{(\mu(i))^2 t}{2\gamma}\right)(\lambda(i))^{1/\gamma}\,\mathrm{d}\nu(i)$$
$$= \left(1+\frac{\hat{\sigma}^2}{\gamma}t\right)^{-1/2}\exp\left(\frac{1}{2}\frac{(\gamma\hat{\mu}+\hat{\sigma}^2 z)^2}{\gamma\hat{\sigma}^2(\gamma+\hat{\sigma}^2 t)} - \frac{1}{2}\frac{\hat{\mu}^2}{\hat{\sigma}^2}\right)$$

By (8), $p(z)$ is equal to a positive multiple of

$$\left(1+\frac{\hat{\sigma}^2}{\gamma}t\right)^{-\gamma/2}\exp\left(\frac{1}{2}\frac{(\gamma\hat{\mu}+\hat{\sigma}^2 z)^2}{\hat{\sigma}^2(\gamma+\hat{\sigma}^2 t)} - \frac{1}{2}\frac{\gamma\hat{\mu}^2}{\hat{\sigma}^2}\right)$$

Since $p = \mathrm{d}P_0/\,\mathrm{d}P$, $(\,\mathrm{d}P_0/\,\mathrm{d}\Lambda)(z)$ is equal to a positive multiple of

$$(2\pi t)^{-1/2}\exp\left(-\frac{z^2}{2t}\right)\left(1+\frac{\hat{\sigma}^2}{\gamma}t\right)^{-\gamma/2}\exp\left(\frac{1}{2}\frac{(\gamma\hat{\mu}+\hat{\sigma}^2 z)^2}{\hat{\sigma}^2(\gamma+\hat{\sigma}^2 t)} - \frac{1}{2}\frac{\gamma\hat{\mu}^2}{\hat{\sigma}^2}\right).$$

Here,

$$\exp\left(-\frac{z^2}{2t}\right)\exp\left(\frac{1}{2}\frac{(\gamma\hat{\mu}+\hat{\sigma}^2 z)^2}{\hat{\sigma}^2(\gamma+\hat{\sigma}^2 t)} - \frac{1}{2}\frac{\gamma\hat{\mu}^2}{\hat{\sigma}^2}\right) = \exp\left(-\frac{1}{2}\frac{(z-\hat{\mu}t)^2}{t(1+(\hat{\sigma}^2/\gamma)t)}\right).$$

Thus, $(\,\mathrm{d}P_0/\,\mathrm{d}\Lambda)(z)$ is equal to a positive multiple of

$$\left(1+\frac{\hat{\sigma}^2}{\gamma}t\right)^{-\gamma/2}(2\pi t)^{-1/2}\exp\left(-\frac{1}{2}\frac{(z-\hat{\mu}t)^2}{t(1+(\hat{\sigma}^2/\gamma)t)}\right)$$
$$= \left(1+\frac{\hat{\sigma}^2}{\gamma}t\right)^{(1-\gamma)/2}\left(2\pi t\left(1+\frac{\hat{\sigma}^2}{\gamma}t\right)\right)^{-1/2}\exp\left(-\frac{1}{2}\frac{(z-\hat{\mu}t)^2}{t(1+(\hat{\sigma}^2/\gamma)t)}\right).$$

Since $(\,\mathrm{d}P_0/\,\mathrm{d}\Lambda)(z)$ integrates to one,

$$\frac{\mathrm{d}P_0}{\mathrm{d}\Lambda}(z) = \left(2\pi t\left(1+\frac{\hat{\sigma}^2}{\gamma}t\right)\right)^{-1/2}\exp\left(-\frac{1}{2}\frac{(z-\hat{\mu}t)^2}{t(1+(\hat{\sigma}^2/\gamma)t)}\right).$$

Thus P_0 is the normal distribution with mean $\hat{\mu}t$ and variance $t(1+(\hat{\sigma}^2/\gamma)t)$. Since $t(1+(\hat{\sigma}^2/\gamma)t) > t$, we conclude that the representative consumer's probabilistic belief is more dispersed than the objective probability measure, and also than any individual consumer's probabilistic belief.

6 Conclusion

In this paper, we have asked the question on how the heterogeneity in individual consumers' subjective beliefs affect the prices for derivative assets. We have provided a condition on a family of distributions, which is satisfied by log-normal, Gamma, and many other families of distributions, that guarantees that heterogeneous beliefs increase the prices for derivative assets as long as all individual consumers' beliefs belongs to that family.

An important direction of future research is to extend the result of this paper to the case of a continuous-time economy, along the lines of Jouini and Napp (2007). Such an extension will allow us to more fully analyze how the discrepancy between the prediction by the Black-Scholes option pricing formula and the options prices observed in real-world markets can be reconciled by introducing heterogeneous beliefs.

A Proofs

In this appendix, we give proofs of the results needed to establish our main results. Assumptions 1, 2, and 3 are met throughout.

Taking the logarithm of both sides of (4) and (5), and then differentiating them, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln\pi(z) = \frac{\pi'(z)}{\pi(z)} = q(z) - \frac{1}{s(c(z))},$$
$$\frac{\mathrm{d}}{\mathrm{d}z}\ln\pi(z|\alpha) = \frac{\frac{\partial\pi}{\partial z}(z|\alpha)}{\pi(z|\alpha)} = q(z|\alpha) - \frac{1}{s(c(z))}$$

for all $z \in Z$. Thus, for all $z^* \in Z$ and $z \in Z$,

$$\frac{\pi(z)}{\pi(z^*)} = \exp\left(\int_{z^*}^{z}\left(q(t) - \frac{1}{s(c(t))}\right)\mathrm{d}t\right),$$
$$\frac{\pi(z|\alpha)}{\pi(z^*|\alpha)} = \exp\left(\int_{z^*}^{z}\left(q(t|\alpha) - \frac{1}{s(c(t))}\right)\mathrm{d}t\right).$$

Thus, together with the definitions of p and $p(\cdot|\alpha)$, we obtain

$$\frac{p(z)/p(z|\alpha)}{p(z^*)/p(z^*|\alpha)} = \frac{\pi(z)/\pi(z|\alpha)}{\pi(z^*)/\pi(z^*|\alpha)} = \exp\left(\int_{z^*}^{z}(m(t)-\alpha)\,g(t)\,\mathrm{d}t\right) \tag{9}$$

for all $z^* \in Z$ and $z \in Z$.

We then obtain the following lemma on the comparison between the "fictitious" probability density $p(\cdot|\alpha)$ and the "true" probability density p, and between the "fictitious" state-price density $\pi(\cdot|\alpha)$ and the "true" state-price density π.

Lemma 1 *Let $\alpha \in \boldsymbol{R}$.*

1. *Let $z^* \in Z$. If $p(z^*) = p(z^*|\alpha)$ and $p'(z^*) \geq \partial p(z^*|\alpha)/\partial z$, then $p(z) > p(z|\alpha)$ for every $z > z^*$.*

2. *Let $z^* \in Z$. If $p(z^*) = p_\alpha(z^*)$ and $p'(z^*) \leq \partial p(z^*|\alpha)/\partial z$, then $p(z) > p(z|\alpha)$ for every $z < z^*$.*

3. *There are no more than two z's such that $p(z) = p(z|\alpha)$.*

4. *If there are exactly two z's, denoted by z_1 and z_2 with $z_1 < z_2$, such that $p(z) = p(z|\alpha)$, then $p(z) < p(z|\alpha)$ whenever $z_1 < z < z_2$; and $p(z) > p(z|\alpha)$ whenever $z < z_1$ or $z > z_2$.*

5. *Let $z^* \in Z$. If $\pi(z^*) = \pi(z^*|\alpha)$ and $\pi'(z^*) \geq \partial\pi(z^*|\alpha)/\partial z$, then $\pi(z) > \pi(z|\alpha)$ for every $z > z^*$.*

6. *Let $z^* \in Z$. If $\pi(z^*) = \pi(z^*|\alpha)$ and $\pi'(z^*) \leq \partial\pi(z^*|\alpha)/\partial z$, then $\pi(z) > \pi(z|\alpha)$ for every $z < z^*$.*

7. *There are no more than two z's such that $\pi(z) = \pi(z|\alpha)$.*

8. *If there are exactly two such z's, denoted by z_1 and z_2 with $z_1 < z_2$, then $\pi(z) < \pi(z|\alpha)$ whenever $z_1 < z < z_2$; and $\pi(z) > \pi(z|\alpha)$ whenever $z < z_1$ or $z > z_2$.*

Proof of Lemma 1 Let z^* be as in the statement of this lemma. Then $q(z^*) \geq q(z^*|\alpha)$ and hence $m(z^*) \geq \alpha$. Since m is strictly increasing, $m(z) > \hat{\alpha}$ for every $z > z^*$. Thus $q(z) > q(z|\alpha)$ for every $z > z^*$. By (9), therefore, $p(z)/p(z|\alpha) > 1$ for every $z > z^*$. This proves Part 1.

Part 2 can be proved analogously, noticing that

$$\int_{z^*}^{z} (m(t) - \alpha)\, g(t)\, \mathrm{d}t = \int_{z}^{z^*} (\alpha - m(t))\, g(t)\, \mathrm{d}t.$$

To prove Part 3, suppose that there are three z's, denoted as $z_1 < z_2 < z_3$ such that $p(z) = p(z|\alpha)$. Then, by applying the contrapositive of Part 1 to $z = z_2$ and $z^* = z_3$, we obtain $p'(z_2) < \partial p(z_2|\alpha)/\partial z$. By applying the contrapositive of Part 2 to $z = z_2$ and $z^* = z_1$, we obtain $p'(z_2) > \partial p(z_2|\alpha)/\partial z$. This is a contradiction. Part 3 is thus proved.

To prove Part 4, apply the contrapositive of Part 1 to $z = z_1$ and $z^* = z_2$, we obtain $p'(z_1) < \partial p(z_1|\alpha)/\partial z$. Similarly, apply the contrapositive of Part 2 to $z = z_2$ and $z^* = z_1$, we obtain $p'(z_2) > \partial p(z_2|\alpha)/\partial z$. These inequalities are sufficient to establish Part 4.

By (9), Parts 5 through 8 can be proved analogously to Parts 1 through 4. ///

The following lemma shows when Parts 4 and 5 of Lemma 1 are applicable.

Lemma 2 *Let $\alpha \in \boldsymbol{R}$.*

1. *If there is a strictly increasing function $k : Z \to \boldsymbol{R}$ such that $E(pk) = E(p(\cdot|\alpha)k)$, then there are exactly two z's such that $\pi(z) = \pi(z|\alpha)$.*

2. *If there is a strictly increasing function $k : Z \to \boldsymbol{R}$ such that $E(\pi k) = E(\pi(\cdot|\alpha)k)$, then there are exactly two z's such that $\pi(z) = \pi(z|\alpha)$.*

Proof of Lemma 2 To prove Part 1, by Part 3 of Lemma 1, it suffices to prove that there are at least two z's such that $p(z) = p(z|\alpha)$. We shall do so by a contradiction argument. Indeed, if there were no such z, then $p(z) > p(z|\alpha)$ for every $z \in Z$ or $p(z) < p(z|\alpha)$ for every $z \in Z$, because p and p_α are continuous. But then either $E(p) > E(p_\alpha)$ or $E(p) < E(p(\cdot|\alpha))$, a contradiction to $E(p) = E(p(\cdot|\alpha)) = 1$. If there were a unique z such that $p(z) = p(z|\alpha)$, denoted by z_0, then the difference $p - p(\cdot|\alpha)$ would change its sign only at z_0. Suppose that $p(z) - p(z|\alpha) < 0$ for every $z < z_0$ and $p(z) - p(z|\alpha) > 0$ for every $z > z_0$. Since k is a strictly increasing function, $(p(z) - p(z|\alpha))(k(z) - k(z_0)) > 0$ for every $z \neq z_0$. Since P is not concentrated on any single point,

$$E((p - p(\cdot|\alpha))(k - k(z_0))) > 0.$$

However, the left-hand side of this inequality is equal to

$$(E(pk) - E(p(\cdot|\alpha)k)) - (E(p) - E(p(\cdot|\alpha)))\,k(z_0) = 0.$$

This is a contradiction. Hence there are exactly two z's such that $p(z) = p_\alpha(z)$.

Part 2 can be established by using the same argument and Part 7 Lemma 1, it ///

Franke, Stapleton, and Subrahmanyam (1999, Lemma 1) also established Part 6, albeit under a much more restrictive condition, that the elasticity of π, $\pi'(z)z/\pi(z)$, is a strictly decreasing function of z, while the elasticity of π_α is constant.

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