Necessary and Sufficient Conditions for the Efficient Risk-Sharing Rules and the Representative Consumer's Utility Function

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Abstract

We show that for every collection of increasing risk-sharing rules and for every increasing and concave expected utility function, there exists a collection of increasing and concave expected utility functions for which the given risk-sharing rules are efficient and the given utility function coincides with the corresponding representative consumer's utility function. We then determine the smallest class of utility functions that contains not only all functions exhibiting constant relative risk aversion but also all functions derivable as the representative consumer's utility function from such utility functions; and also fully characterize the efficient risk-sharing rules in this class. Furthermore, we show that in a two-consumer economy, assuming that the two have the same utility function imposes no additional restriction on the efficient risk-sharing rules.

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1 Introduction

We consider an economy under uncertainty with a single consumption good in which all consumers have expected utility functions with respect to a homogeneous probabilistic belief and, should there be more than one consumption periods, a common discount rate, but their utility functions can be different. As usual, we assume that all consumers prefer more to less and are averse to risk, which means that their utility functions are increasing and concave. Then, a

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Pareto-efficient allocation can be described by a collection of efficient *risk-sharing rules*, one for each consumer, and its supporting (or, decentralizing) price system can be represented by the utility function for the *representative consumer*. Specifically, an individual consumer's risk-sharing rule is a deterministic function that maps each realized aggregate consumption level to a consumption level for the individual consumer; and the representative consumer's marginal utility function serves as a *pricing kernel*, in the sense that the price of an asset is the sum of all its future dividends multiplied by the stochastic marginal rates of substitution between the current and future time at which the dividends is paid.

A couple of properties are well known for risk-sharing rules and the representative consumer's utility function. First, every individual consumer's efficient risk-sharing rule is an increasing function. This property is called the *comonotonicity* because it is equivalent to the property that the consumers' consumption levels are increasing functions of one another. Second, the representative consumer's utility function is an increasing and concave function of aggregate consumption levels. This is equivalent to saying that the pricing kernel is a positive and decreasing function of aggregate consumption levels. These properties hold for every efficient allocation regardless of the distribution of stochastic aggregate endowments (or stochastic processes of aggregate endowments) and coefficients of risk aversion of individual consumers.

The benchmark result for this paper (Theorem 2) is that the efficiency implies no property other than the comonotonicity of the risk-sharing rules and the monotonicity and risk aversion of the representative consumer's utility function. More specifically, for every collection of increasing risk-sharing rules, one for each consumer, and for every increasing and concave utility function, there exists a collection of increasing and concave utility functions for which the given risk-sharing rules are efficient and the given utility function coincides with the corresponding representative consumer's utility function. We also show that each individual consumer's utility function are uniquely determined up to scalar addition by the risk-sharing rules and the representative consumer's utility function. Versions of this result have been known in the literature, and we compare our version with the exisiting ones.

In the rest of this paper, we investigate how robust the benchmark result is by restricting the collection of individual consumers' utility functions in two directions. One is to narrow down the class of utility functions, by requiring that all consumers exhibit constant relative risk aversion (CRRA), albeit at different levels. The other is to narrow down the class of distributions of utility functions, by requiring that all consumers have the same utility function. The choice of these two directions is motivated by the celebrated mutual fund theorem. The theorem states that if all consumers exhibit CRRA, and if their utility functions are the same, then the efficient risk-sharing rules are linear and the representative consumer has the same CRRA utility function as the individual consumers. We would therefore like to investigate what kind of risk-sharing rules and utility functions can possibly emerge if either one of the two assumptions of the theorem is dispensed with.

First, we identify the smallest class of utility functions, which we denote by $\mathscr{W}$, that contain all utility functions exhibiting constant relative risk aversion and all utility functions that can be derived as the reprsentative consumer's utility function from any collection of CRRA utility functions (Propositions 2). The CRRA utility functions, such as power and log functions, have been most commonly used in the literature, especially of macroeconomics and finance, but a drawback of this class of utility functions is that even if all consumers in the economy have CRRA utility functions, the representative consumer of the economy does not have any CRRA utility function, unless all consumers' utility functions are the same. By identifying the class $\mathscr{W}$ of all utility functions derivable for the representative consumer from CRRA utility functions, we identify the appropriate class of utility functions for asset pricing within the class of time-additive expected utility functions.

In the class $\mathscr{W}$, we also characterize (Proposition 3) the efficient risk-sharing rules in the following two-step manner. First, we characterize, as the inverse of the sum of power functions, the efficient risk-sharing rule in an economy populated by consumers having CRRA utility functions. Second, we prove that in an economy of utility functions in $\mathscr{W}$, each consumer's risk-sharing rule can be represented as a weighted sum of risk-sharing rules of consumers having CRRA utility functions. This result extends the mutual fund theorem and generalizes some of the results of Section 6 of Hara, Huang, and Kuzmics (2007). Based on this result, we also show (Corollary 1) that if all consumers have CRRA utility functions, then the risk-sharing rules are essentially uniquely determined by the utility function for the representative consumer. This is consistent and yet ought to be contrasted with the benchmark theorem (Theorem 2): with the restriction to CRRA utility functions, a pricing kernel not only narrows down the class of compatible risk-sharing rules but essentially uniquely determines the risk-sharing rules.

Second, we consider the case in which all consumers have the same utility function. This case is of special interest because the assumption of identical utility functions has been commonly used under the name of *ex ante homogeneity* in the literature of dynamic macroeconomics, such as Weil (1992) and Krussel and Smith (1998). An important property of efficient risk-sharing rules in this case, which was exploited by Mazzocco and Saini (2006) in their hypothesis testing, is that if one consumer consumes more than another at some aggregate consumption level, then the former must necessarily consume more than the latter at every aggregate consumption level. We show that in a two-consumer economy, the assumption of identical utility functions does not impose any other additional restriction on the efficient risk-sharing rules. However, we show, by means of an example, that if there are more than two consumers, then there are some additional restrictions on the efficient-risk sharing rules.

These results are established for the case in which the feasible consumption levels are strictly positive numbers, that is, the domain of a utility function is $\boldsymbol{R}_{++}$. We show in Section 7 that some of these results can be generalized to the case in which the domain of a utility function is an arbitrary open interval of $\boldsymbol{R}$. Such generalizations are important, not just to satisfy theoretical curiousity. For example, both Townsend (1994) and Ogaki and Zhang (2001) used utility functions exhibiting hyperbolic absolute risk aversion in their tests of the full-insurance hypothesis. The domain of utility functions of Townsend (1994) is $\boldsymbol{R}_{++}$ and hence the utility functions must necessarily exhibit contant relative risk aversion, while the domains of utility functions of Ogaki and Zhang (2001) are open intervals of $\boldsymbol{R}$ and hence the utility functions can exhibit decreasing and increasing relative risk aversion. The full-insurance hypothesis is rejected by the former but not by the latter. The sensitivity of empirical results with respect to the choice of domains suggests that our rsults should be generalized to utility functions of arbitrary domains.

This paper is organized as following. Section 2 lays out the setting and formulates our problem. Section 3 states the benchmark result (Theorem 2) and its relationship with exisitng results. Section 4 states and proves the benchmark theorem in terms of marginal utility functions, or the pricing kernel. Section 5 deals with the newly proposed class $\mathscr{W}$, obtained from aggregating CRRA utility functions. Section 6 deals with the case where the consumers have the same utility function. Section **??** gives analytical examples of implications of the main results. Section 7 shows how these results can be generalized to the case in which the domains of utility functions are arbitrary. Section 8 gives concluding remarks and suggests directions of future research.

2 Setting

There are I consumers, $i \in \{1, \dots, I\}$. Consumer i has a von-Neumann Morgenstern utility function (also known as Bernoulli utility function) $u_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$, where $\boldsymbol{R}_{++}$ is the set of all real numbers strictly greater than 0. We assume that u_i is of class C^r with $r \geq 2$, $r = \infty$ (in which case, u_i is infinitely many times differentiable), or $r = \omega$ (in which case, u_i is real analytic), and satisfies $u_i'(x_i) > 0$ and $u_i''(x_i) < 0$ for every $x_i \in \boldsymbol{R}_{++}$. We also assume that u_i satisfies the Inada condition, that is, $u_i'(x_i) \to \infty$ as $x_i \to 0$ and $u_i'(x_i) \to 0$ as $x_i \to \infty$.

Imagine that the consumers have common probabilistic beliefs on the uncertainty of the economy and their preferences over state-contingent consumption plans c^i are given by the expected utility functions $E\left(u_i(c^i)\right)$. It is well known that each Pareto-efficient allocation of an aggregate consumption plan c can be characterized as a solution to the welfare maximization problem

$$\begin{aligned} \max_{(c^1,\dots,c^I)} \quad & \sum \lambda_i E\left(u_i(c^i)\right), \\ \text{subject to} \quad & \sum c^i = c, \end{aligned} \tag{1}$$

for some $\lambda = (\lambda_1, \dots, \lambda_I) \in \boldsymbol{R}^I_{++}$. Since both the objective and constraint functions are separable state by state, we can reduce this problem to the following one, for each realized aggregate consumption level $x \in \boldsymbol{R}_{++}$, involving realized consumption levels, not state-contingent consumption plans:

$$\begin{aligned} \max_{(x_1,\dots,x_I) \in \boldsymbol{R}_{++} \times \cdots \times \boldsymbol{R}_{++}} \quad & \sum \lambda_i u_i(x_i), \\ \text{subject to} \quad & \sum x_i = x. \end{aligned} \tag{2}$$

By the strict concavity and the Inada condition, for each x, there exists exactly one solution to this problem. We now denote it by $f(x) = (f_1(x), \dots, f_I(x))$. Then the function $f : \boldsymbol{R}_{++} \to \boldsymbol{R}^I_{++}$ is well defined. This is the *risk-sharing rule*, which maps each aggregate consumption level to a profile of the individual consumers' consumption levels. By a slight abuse of terminology, we will also refer to each coordinate function f_i as a risk-sharing rule. It follows from the first-order condition and the implicit function theorem that f is of class C^{r-1}, where $\infty - 1 = \infty$ and $\omega - 1 = \omega$ by convention, and that $f_i'(x) > 0$ for every $x \in \boldsymbol{R}_{++}$, $f_i(x) \to 0$ as $x \to 0$, and $f_i(x) \to \infty$ as $x \to \infty$, for every i. Define $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ as the value function of this problem; that is, $u(x) = \sum_i \lambda_i u_i(f_i(x))$. This is the *representative consumer's utility function.* By the envelope theorem, $u'(x) = \lambda_i u_i'(f_i(x))$ for every i and every $x \in \boldsymbol{R}_{++}$. This shows that $u'(x) > 0$, u satisfies the Inada condition, and u' is of class C^{r-1}. Hence u is of class C^r and $u''(x) = \lambda_i u_i''(f_i(x)) f_i'(x) < 0$. These properties have been well known in the literature.

The importance of the risk-sharing rule f and the representative consumer's utility function u lies in the following facts. Under appropriate integrability conditions, an allocation $(c_1, \dots, c_I)$ of state-contingent consumption plans is a Pareto-efficient allocation of an aggregate consumption plan c if and only if $(c_1, \dots, c_I) = (f_1(c), \dots, f_I(c))$ for some $\lambda \in \boldsymbol{R}^I_{++}$. Moreover, then, $u'(c)$ is the unique pricing kernel, modulo strictly positive scalar multiplications, that supports (decentralizes) c.

Let us make two remarks here. First, while we set up the original welfare maximization problem (1) in a static model with a single consumption period and then reduced it to (2), we can still reduce the original welfare maximization problem to (2) in a dynamic model of discrete or continuous time if, in addition, the consumers share a common time-discount rate. The details on how to do so are given in Hara (2006, Section 3). Second, the above characterization of an efficient allocation of an aggregate consumption plan in terms of the original and reduced welfare maximization problems ((1) and (2)) is valid not only in an exchange economy but also in a production economy. The only difference between the two cases is that c must be equal to the exogenously given aggregate endowment in an exchange economy, while it is endogenously determined at equilibrium in a production economy.

The argument so far can be more succinctly stated as follows.

For each $r \in \{2, 3, \dots, \infty, \omega\}$, let $\mathscr{U}_r$ be the set of all C^r functions $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ that satisfy $u'(\boldsymbol{R}_{++}) = \boldsymbol{R}_{++}$ and $u''(\boldsymbol{R}_{++}) \subseteq -\boldsymbol{R}_{++}$. For each $I \in \{1, 2, \dots\}$ and each $r \in \{2, 3, \dots, \infty, \omega\}$, let $\mathscr{F}^I_r$ be the set of all C^r functions $f = (f_1, \dots, f_I) : \boldsymbol{R}_{++} \to \boldsymbol{R}^I_{++}$, with $f_i : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ for every i, such that $f_i(\boldsymbol{R}_{++}) = \boldsymbol{R}_{++}$ and $f_i'(\boldsymbol{R}_{++}) \subseteq \boldsymbol{R}_{++}$ for every i, and $\sum_i f_i = \chi$, where χ is the identity function on $\boldsymbol{R}_{++}$. To simplify exposition, we often write $\mathscr{U}$ for $\mathscr{U}_2$, and $\mathscr{F}^I$ for $\mathscr{F}^I_1$.

Now consider the following welfare maximization problem:

$$\begin{array}{cc} \max\limits_{(x_1,\dots,x_I) \in \boldsymbol{R}_{++} \times \dots \times \boldsymbol{R}_{++}} & \sum u_i(x_i), \\ \text{subject to} & \sum x_i = x. \end{array} \tag{WMP}$$

This problem seems to be a special case of (2) as we obtain the former from the latter by letting $\lambda_i = 1$ for every i. But in fact the former can deal with all the cases that the latter can deal with because, even when $\lambda_i \neq 1$ for some i, (2) is nothing but (WMP) with $\lambda_i u_i$ taking the place of u_i for each i. We say that $f : \boldsymbol{R}_{++} \to \boldsymbol{R}^I_{++}$ and $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ are *derived from* $(u_1, \dots, u_I)$ *via* *the welfare maximization problem* (WMP for short) if $f(x)$ is a solution to (WMP) for every $x \in \boldsymbol{R}_{++}$ and u is the value function of (WMP).

What we have stated above can then be restated as follows:

Theorem 1 *For every $I \in \{1, 2, \dots\}$, every $r \in \{2, 3, \dots, \infty, \omega\}$, and every $(u_1, \dots, u_I) \in \mathscr{U}_r^I$, if $f : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}^I$ and $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ are derived from $(u_1, \dots, u_I)$ via WMP, then $f \in \mathscr{F}_{r-1}^I$ and $u \in \mathscr{U}_r$.*

3 Benchmark Theorem

The benchmark theorem for this paper is the converse of Theorem 1:

Theorem 2 *For every $I \in \{1, 2, \dots\}$, every $r \in \{2, 3, \dots, \infty, \omega\}$, every $f \in \mathscr{F}_{r-1}^I$, and every $u \in \mathscr{U}_r$, there exists a $(u_1, \dots, u_I) \in \mathscr{U}_r^I$ such that f and u are derived from $(u_1, \dots, u_I)$ via WMP. Moreover, for every $(v_1, \dots, v_I) \in \mathscr{U}^I$, if f and u are derived from $(v_1, \dots, v_I)$ via WMP, then $u_i - v_i$ is constant for every i, that is, for every i, there exists a $k_i \in \boldsymbol{R}$ such that $u_i(x_i) - v_i(x_i) = k_i$ for every $x_i \in \boldsymbol{R}_{++}$.*

The first part of this theorem is the converse of Theorem 1, which establishes the existence of a profile $(u_1, \dots, u_I) \in \mathscr{U}_r^I$ of individual consumers' utility functions that is consistent with the given risk-sharing rule f and the given utility function u for the representative consumer. The second part is the uniqueness of the profile up to scalar additions. There is no degree of freedom with respect to scalar multiplication, as it would typically affect u when derived via WMP. Since adding a constant to a utility function does not change the risk attitude it represents, this theorem implies that f and u uniquely determines each individual consumer's risk attitudes. The proof of this theorem is given in the next section.

Let's now compare this result with existing ones. First, Dana and Meilijson (2003) also constructed a collection of increasing and concave utility functions for which the given risk-sharing rules are efficient and the given utility function coincides with the corresponding representative consumer's utility function. There are two important properties for the utility functions that they did not guarantee but we do here.[1] One is the Inada condition, which says that the marginal utility spans from zero to infinity. The other is the differentiability of arbitrary order. Although the existence of an interior optimal consumption is crucial for many applications, it would not be guaranteed without the Inada condition. The curvature of risk-sharing rules, which can be used to check the validity of the mutual fund theorem, cannot even be defined without four times differentiability of utility functions. Our proof method depends on the envelope theorem for the welfare maximization problem that determines the risk-sharing rules and the utility function for the representative consumer. This method allows us to easily relate them to the individual consumers' utility functions and thus explore many applications.

[Mazzocco and Saini and Kubler here.]

[1] To do so, we need to impose some extra conditions on the given risk-sharing rules and the given utility function for the representative consumer.

What can we learn from the benchmark theorem (Theorem 2)? First, it shows that the comonotonicity of risk-sharing rules and the monotonicity and risk aversion of the representative consumer's utility function exhaust all the implications of efficiency, when the consumers have expected utility functions with respect to a homogeneous probabilistic belief and a common discount rate, but no other condition is imposed on their utility functions. The benchmark theorem also shows the essential uniqueness of the individual utility functions once a collection of risk-sharing rules and a utility function for the representative consumer are given. It should be emphasized that this uniqueness result is no less important than the existence result, because it allows us to pin down the nature of biases in the inference of individual utility functions when a "wrong" utility function is postulated for the representative consumer.

Second, but perhaps more important to empirical studies on risk-sharing rules, since a collection of risk-sharing rules and a utility function for the representative consumer can be given independently of each other, either does not give any information on the other. In particular, linearity the risk-sharing rules does not automatically imply that the consumers exhibit contant relative risk aversion.

We now substantiate this claim by giving an example involving risk-sharing rules that are linear, a conclusion of the mutual fund theorem. Let $(\theta_1, \ldots, \theta_I) \in \boldsymbol{R}^I_{++}$ satisfy $\sum_i \theta_i = 1$. Define $f \in \mathscr{F}^I$ by letting $f_i(x) = \theta_i x$ for every i and $x \in \boldsymbol{R}_{++}$. Let $u \in \mathscr{U}$ and denote its relative risk aversion by $b : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$. By Theorem 2, there exists a $(u_1, \ldots, u_I) \in \mathscr{U}^I$ such that f and u are derived from $(u_1, \ldots, u_I)$ via WMP. Denote the relative risk aversion of u_i by b_i. By the first-order condition for the solution to WMP,

$$u'(x) = u'_i\left(f_i(x)\right) = u'_i\left(\theta_i x\right) \tag{3}$$

for every $x \in \boldsymbol{R}_{++}$. By differentiating both sides of (19) with respect to x, we obtain

$$u''(x) = u''_i(\theta_i x)\theta_i \tag{4}$$

Divide both sides of (4) by their counterparts of (19) and multiply $-x$, then we obtain

$$b(x) = b_i(\theta_i x) \tag{5}$$

for every i and $x \in \boldsymbol{R}_{++}$. This equality shows that u exhibits constant relative risk aversion if and only if every u_i exhibits constant relative risk aversion. This is nothing but the case dealt with by the mutual fund theorem. However, every u_i exhibits decreasing relative risk aversion whenever so does u, and every u_i exhibits increasing relative risk aversion whenever so does u. Since u can be arbitrarily chosen, this result implies that there are more than one profile of individual consumers' utility functions that are consistent with a given profile of linear risk-sharing rules, and these profiles can consist of utility functions exhibiting decreasing relative risk aversion or of utility functions exhibiting increasing relative risk aversion.

Friend and Blume (1975) used a data set on households' wealth levels and asset allocation to estimate the proportion of wealth invested into risky assets. The result depended on how to treat housing in the calculation of total wealth, but the choice between different treatments only had quantitatively minor impact on their estimates. Their overall finding is that the proportion invested into risky assets does not greatly depend on wealth level. Then they concluded that it is reasonable to assume that the consumers (households) exhibit constant relative risk aversion. If the consumers were to implement the consumption allocation resulting from linear risk-sharing rules, then each must hold a fraction of the market portfolio, consisting of all assets in the economy. Then the proportion of wealth they invest into each asset must be common across them. In particular, the linearity of the risk-sharing rules implies that the proportion of the risky assets in total wealth is independent of wealth levels, as empirically found by Friend and Blume (1975). Our benchmark theorem, however, implies that the linear risk-sharing rules are compatible with increasing or decreasing relative risk aversion for both the representative and individual consumers. Friend and Blume's (1975) conclusion that it is reasonable to assume that they exhibit constant relative risk aversion is, therefore, not theoretically warranted.

A related strand of literature is on the empirical testing of the full insurance hypothesis. Townsend (1994), Mace (1991), Cochrane (1991), and Kohara, Ohtake, and Saito (2002) conducted tests for efficiency for the case, among others, in which all consumers (or households) have equal and constant relative risk aversion. In this case, all the efficient risk-sharing rules are linear and hence the hypothesis of an efficient allocation is rejected whenever the observed data set is inconsistent with linearity. This, in fact, produces tests both for efficiency and functional forms of utility functions togther, and the joint hypothesis is often rejected in the literature.

The tests of Ogaki and Zhang (2001) relaxed the assumption of the common constant relative risk aversion to the assumption of hyperbolic absolute risk aversion with a common cautiousness,[2] in order to accommodate the possibility of decreasing relative risk aversion. In this case, the risk-sharing rules are affine, though not necessarily linear,[3] and the hypothesis of an efficient allocation is often not rejected. According to the benchmark theorem, however, this result in no way supports decreasing relative risk aversion, as there are even utility functions exhibiting increasing relative risk aversion that generate the observed affine risk-sharing rules.

Kubler (2003)

Altonji, Hayashi, and Kotlikoff?

4 Pricing Kernel

When it comes to proving the benchmark theorem (Theorem 2) and other results, it is more convenient to deal with marginal utility functions, or pricing kernels, rather than the utility functions themselves. We shall now provide an alternative formulation of the risk-sharing rules and the representative consumer's utility function in terms of marginal utility functions.

For each $r \in \{1, 2, \ldots, \infty, \omega\}$, let $\mathscr{P}_r$ be the set of all C^r functions $p : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ that

[2]The cautiousness is the derivative of the reciprocal of the absolute risk aversion. In the case of hyperbolic absolute risk aversion, it is constant.

[3]In this paper, we distinguish affinity and linearity. The graph of a linear function must go through the origin, while the graph of an affine function need not.

satisfy $p(\boldsymbol{R}_{++}) = \boldsymbol{R}_{++}$ and $p'(\boldsymbol{R}_{++}) \subseteq -\boldsymbol{R}_{++}$. To simplify exposition, we often write $\mathscr{P}$ for $\mathscr{P}_1$.

We say that $f : \boldsymbol{R}_{++} \to \boldsymbol{R}^I_{++}$ and $p : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ are *derived from* $(p_1, \dots, p_I)$ *via first-order conditions* (FOC for short) if $p^{-1} = \sum_i p_i^{-1}$ and $f_i = p_i^{-1} \circ p$ for every i.[4] It is easy to check that for every $I \in \{1, 2, \dots\}$ and every $r \in \{1, 2, \dots, \infty, \omega\}$, if $(p_1, \dots, p_I) \in \mathscr{P}^I_r$, and f and p are derived from $(p_1, \dots, p_I)$ via FOC, then $f \in \mathscr{F}^I_r$ and $p \in \mathscr{P}_r$. The converse can also be proved quite easily.

Theorem 3 *For every $I \in \{1, 2, \dots\}$, every $r \in \{1, 2, \dots, \infty, \omega\}$, every $f \in \mathscr{F}^I_r$, and every $p \in \mathscr{P}_r$, there exists a unique $(p_1, \dots, p_I) \in \mathscr{P}^I_r$ such that f and p are derived from $(p_1, \dots, p_I)$ via FOC.*

Proof of Theorem 3 For each i, define $p_i : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $p_i = p \circ f_i^{-1}$, then it is easy to show that $f_i = p_i^{-1} \circ p$ for every i and $p_i \in \mathscr{P}_r$. Adding this over i, we see that $\sum_i \left(p_i^{-1} \circ p\right) = \left(\sum_i p_i^{-1}\right) \circ p = \chi$. Hence $p^{-1} = \sum_i p_i^{-1}$. The uniqueness follows from the fact that $p = p_i \circ f_i$ if and only if $p_i = p \circ f_i^{-1}$. ///

The relationship between $\mathscr{P}_r$ and $\mathscr{U}_{r+1}$ is straightforward: if $u \in \mathscr{U}_{r+1}$, then $u' \in \mathscr{P}_r$. Conversely, if $p \in \mathscr{P}_r$, then any particular integral of p belongs to $\mathscr{U}_{r+1}$. The relationship between WMP and FOC is as follows.

Lemma 1 *Let $I \in \{1, 2, \dots\}$ and $r \in \{2, 3, \dots, \infty, \omega\}$.*

1. *Let $(u_1, \dots, u_I) \in \mathscr{U}_r$, $u \in \mathscr{U}_r$, and $f \in \mathscr{F}^I_{r-1}$. If f and u are derived from $(u_1, \dots, u_I)$ via WMP, then f and u' are derived from $(u'_1, \dots, u'_I)$ via FOC.*
2. *Let $(p_1, \dots, p_I) \in \mathscr{P}_{r-1}$, $p \in \mathscr{P}_{r-1}$, and $f \in \mathscr{F}^I_{r-1}$. If f and p are derived from $(p_1, \dots, p_I)$ via FOC, and if u is a particular integral of p, then for every i there is a particular integral u_i of p_i such that f and u are derived from $(u_1, \dots, u_I)$ via WMP.*

Proof of Lemma 1 1. If f and u are derived from $(u_1, \dots, u_I)$ via WMP, then $u' = u'_i \circ f_i$ by the envelope theorem, and hence $f_i = (u'_i)^{-1} \circ u'$. Moreover, since $\sum_i f_i = \sum_i \left((u'_i)^{-1} \circ u'\right) = \left(\sum_i (u'_i)^{-1}\right) \circ u'$ is the identity function, $\sum_i (u'_i)^{-1} = (u')^{-1}$. Therefore, f and u' are derived from $(u'_1, \dots, u'_I)$ via FOC.

2. Suppose that f and p are derived from $(p_1, \dots, p_I)$ via FOC. For each i, let $\hat{u}_i$ be a particular integral of p_i. Then $f_i = (\hat{u}'_i)^{-1} \circ p$ and hence $p(x) = \hat{u}'_1(f_1(x)) = \cdots = \hat{u}'_I(f_I(x))$ for every $x \in \boldsymbol{R}_{++}$, which implies that f gives the solution to (WMP). Moreover, its value function is equal to one of its particular integral of p, which we denote by u. If $u(x) - \sum_i \hat{u}_i(f_i(x)) = 0$, then we can complete the proof by letting $u_i = \hat{u}_i$. If not, we let $\delta = u(x) - \sum_i \hat{u}_i(f_i(x))$ and define u_i by letting $u_i(x_i) = \hat{u}_i(x_i) + \delta/I$. Then f and u are derived from $(u_1, \dots, u_I)$ via WMP. ///

[4] These relations were used in Karatzas and Shreve (1998, Sections 4.4 and 4.5) for the analysis of utility maximization and equilibrium.

Theorem 2 can now be proved based on Theorem 3 and Lemma 1.

Proof of Theorem 2 Let $f \in \mathscr{F}^I_{r-1}$ and $u \in \mathscr{U}_r$. Then $u' \in \mathscr{P}_{r-1}$ and hence, by Theorem 3, there exists a unique $(p_1, \dots, p_I) \in \mathscr{P}^I_{r-1}$ from which f and u' are derived via FOC. Thus, by part 2 of Lemma 1, for each i, there is a particular integral $u_i \in \mathscr{U}_r$ such that f and u are derived from $(u_1, \dots, u_I)$ via WMP.

As for the uniqueness up to scalar addition, assume that f and u are derived also from $(v_1, \dots, v_I) \in \mathscr{U}^I$ via WMP. Then, by part 1 of Lemma 1, f and u' are derived also from $(v'_1, \dots, v'_I) \in \mathscr{P}^I$ via FOC. By the uniqueness result of Theorem 3, $v'_i = p_i = u'_i$ for every i. Thus $u_i - v_i$ is constant for every i. ///

5 Constant Relative Risk Aversion

In this section, we consider the problem on what kind of restrictions will be imposed on the efficient risk-sharing rules and the representative consumer's utility function by assuming that all consumers exhibit constant relative risk aversion.

Formally, for a utility function $u \in \mathscr{U}$, the relative risk aversion $b : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ is defined by $b(x) = -u''(x)x/u'(x)$ for every $x \in \boldsymbol{R}_{++}$. For a marginal utility function $p \in \mathscr{P}$, the relative risk aversion $b : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ is defined by $b(x) = -p'(x)x/p(x)$ for every $x \in \boldsymbol{R}_{++}$. If b is constantly equal to a $\beta \in \boldsymbol{R}_{++}$, then there exist a $\lambda \in \boldsymbol{R}_{++}$ and a $\gamma \in \boldsymbol{R}_{++}$ such that

$$u(x) = \begin{cases} \lambda \log x + \gamma & \text{if } \beta = 1, \\ \lambda \dfrac{x^{1-\beta} - 1}{1 - \beta} + \gamma & \text{otherwise,} \end{cases}$$

or equivalently,

$$p(x) = \lambda x^{-\beta}$$

for every $x \in \boldsymbol{R}_{++}$. We denote the set of all these $u \in \mathscr{U}$ by $\mathscr{V}$, and the set of all these $p \in \mathscr{P}$ by $\mathscr{Q}$.

Let $I \in \{1, 2, \dots\}$ and for each $i = 1, \dots, I$, let $p_i \in \mathscr{Q}$ satisfy

$$p_i(x_i) = \lambda_i x_i^{-\beta_i}$$

for every $x_i \in \boldsymbol{R}_{++}$. If $p \in \mathscr{P}$ is derived from $(p_1, \dots, p_I)$ via FOC, then

$$p^{-1}(z) = \sum_{i=1}^{I} p_i^{-1}(z) = \sum_{i=1}^{I} \lambda_i^{1/\beta_i} z^{-1/\beta_i}$$

for every $z \in \boldsymbol{R}_{++}$. So we denote by $\mathscr{R}$ the set of all $p \in \mathscr{P}$ for which there exist an $N \in \{1, 2, \dots\}$, a $(c_1, \dots, c_N) \in \boldsymbol{R}^N_{++}$, and a $(s_1, \dots, s_N) \in \boldsymbol{R}^N_{++}$ such that

$$p^{-1}(z) = \sum_{n=1}^{N} c_n z^{-s_n} \tag{6}$$

for every $z \in \boldsymbol{R}_{++}$. Then $\mathscr{R} \supset \mathscr{Q}$, and if $(p_1, \dots, p_I) \in \mathscr{Q}^I$ and p is derived from $(p_1, \dots, p_I)$ via FOC, then $p \in \mathscr{R}$. It can also be shown that if $(p_1, \dots, p_I) \in \mathscr{R}^I$ and p is derived from $(p_1, \dots, p_I)$ via FOC, then $p \in \mathscr{R}$. That is, the set $\mathscr{R}$ is *closed under aggregation.* We can furthermore establish the following result.

Proposition 1 *The set $\mathscr{R}$ is the smallest subset of $\mathscr{P}$ that includes $\mathscr{Q}$ and is closed under aggregation.*

Proof of Proposition 1 It remains to show that for every $p \in \mathscr{R}$, there exist an $I \in \{1, 2, \dots\}$ and a $(p_1, \dots, p_I) \in \mathscr{Q}^I$ such that p is derived from $(p_1, \dots, p_I)$ via FOC. Indeed, if $p \in \mathscr{R}$ is defined by (6), then let $I = N$ and, for each n, let $p_n \in \mathscr{Q}$ be defined by $p_n(x_n) = c_n^{1/s_n} x_n^{-1/s_n}$ for every $x_n \in \boldsymbol{R}_{++}$. Then $p^{-1} = \sum_{n=1}^N p_n^{-1}$. ///

Denote by $\mathscr{W}$ the set of all $u \in \mathscr{U}$ such that $u' \in \mathscr{R}$. Then we can analogously establish the following result

Proposition 2 *The set $\mathscr{W}$ is the smallest subset of $\mathscr{U}$ that includes $\mathscr{V}$ and is closed under aggregation.*

The set $\mathscr{W}$ is nothing but the set of all utility functions $u \in \mathscr{U}$ for which there exist an $N \in \{1, 2, \dots\}$, a $(c_1, \dots, c_N) \in \boldsymbol{R}_{++}^N$, and a $(s_1, \dots, s_N) \in \boldsymbol{R}_{++}^N$ such that

$$\left(u'\right)^{-1}(z) = \sum_{n=1}^{N} c_n z^{-s_n} \tag{7}$$

for every $z \in \boldsymbol{R}_{++}$. The set $\mathscr{W}$ is a more reasonable class of utility functions for the representative consumer than $\mathscr{V}$, for the following reasons. First, it includes $\mathscr{V}$, so that it is sufficiently rich to contain all utility functions exhibiting constant relative risk aversion. Second, it is closed under aggregation, so that the utility functions derived from any utility functions in this class via WMP also belong to this class. This is the property that is missed in the class $\mathscr{V}$ of utility functions exhibiting constant relative risk aversion. Third, this class is the smallest class having these two properties, so that it is the most tractable class of utility functions that are closed under aggregation and admit all levels of constant relative risk aversion.

Next, we characterize the efficient risk-sharing rules when all consumers' marginal utility functions belong to $\mathscr{R}$ (that is, when their utility functions belong to $\mathscr{W}$). To do so, note first that since there are finitely many consumers and each marginal utility function in $\mathscr{R}$ is the sum of finitely many power functions, if $(p_1, \dots, p_I) \in \mathscr{R}^I$, then there are an $N \in \{1, 2, \dots\}$, a $(s_1, \dots, s_N) \in \boldsymbol{R}_{++}^N$, and, for each i, a $(c_{1i}, \dots, c_{Ni}) \in \boldsymbol{R}_{+}^N$ such that

$$p_i^{-1}(z) = \sum_{n=1}^{N} c_{ni} z^{-s_n} \tag{8}$$

for every i and $z \in \boldsymbol{R}_{++}$. Moreover, we can choose them so that $\sum_i c_{ni} > 0$ for every n and $s_n \neq s_m$ whenever $n \neq m$. Keeping this fact in mind, we can state our characterization result as follows.

Proposition 3 *Let $I \in \{1,2,\dots\}$. Let $(p_1,\dots,p_I) \in \mathscr{R}^I$ be given by (8) and $f=(f_1,\dots,f_I) \in \mathscr{F}^I$ be derived from $(p_1,\dots,p_I)$ via FOC. For each n, write $c_n = \sum_i c_{ni}$ and define $g_n : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by*

$$g_n^{-1}(z) = \sum_{m=1}^{N} \frac{c_m}{c_n^{s_m/s_n}} z^{s_m/s_n} \tag{9}$$

for every $z \in \boldsymbol{R}_{++}$. Then

$$f_i(x) = \sum_{n=1}^{N} \frac{c_{ni}}{c_n} g_n(x) \tag{10}$$

for every i and every $x \in \boldsymbol{R}_{++}$.

This proposition can be best understood by first looking at the case in which $s_1,\dots,s_N$ are all distinct, $I=N$, and $c_{ni}>0$ if and only if $n=i$. This means that all consumers exhibit constant relative risk aversion but the levels are all distinct. Then c_{ni}/c_n is equal to 1 if $n=i$ and 0 otherwise. Hence $f_i=g_i$ for every i. This means that $g_1,\dots,g_N$ are the efficient risk-sharing rules in the elementary case in which all consumers exhibit constant, yet distinct, levels of relative risk aversion. Then, in the general case in which the consumers' marginal utility functions do not exhibit constant relative risk aversion, (23) shows that the efficient risk-sharing rules are weighted sums of the efficient risk-sharing rules in the elementary case. Note, however, that even if two consumers have the same risk attitude, their risk-sharing rules need not be any scalar multiples of each other. To see this point, suppose that for some two consumers i and j, there exists a $\mu \neq 1$ such that $p_i(x_i) = \mu p_j(x_j)$ for every $x_i \in \boldsymbol{R}_{++}$. Then $p_i^{-1}(z) = p_j^{-1}(z/\mu)$ and hence

$$\sum_n c_{ni} z^{-s_n} = \sum_n \frac{c_{nj}}{\mu^{s_n}} z^{-s_n}$$

for every $z \in \boldsymbol{R}_{++}$. Thus $c_{ni} = c_{nj}/\mu^{s_n}$ for every n. Since $\mu \neq 1$, this implies that the vector of weights of f_i, $(c_{1i}/c_1,\dots,c_{Ni}/c_N)$, is not a scalar multiple of the vector of weights of f_j, $(c_{1j}/c_1,\dots,c_{Nj}/c_N)$. Since, as can be easily verified, any of $g_1,\dots,g_N$ is not a scalar multiple of another, we can conclude that f_i is not a scalar multiple of f_j.

Proof of Proposition 3 Define $p \in \mathscr{R}$ by $p^{-1} = \sum_i p_i^{-1}$. For each n, define $q_n \in \mathscr{Q}$ by $q_n^{-1}(z) = c_n z^{-s_n}$. Then $g=(g_1,\dots,g_N)$ and p are derived from $(q_1,\dots,q_N)$ via FOC. Indeed,

$$p^{-1}(z) = \sum_{n=1}^{N} c_n z^{-s_n} = \sum_{n=1}^{N} q_n^{-1}(z)$$

and

$$p^{-1}\left(q_n(z)\right) = \sum_{m=1}^{N} c_m \left(\left(\frac{z}{c_n}\right)^{-1/s_n} \right)^{-s_m} = g_n^{-1}(z)$$

for every $z \in \boldsymbol{R}_{++}$, implying that $g_n = q_n^{-1} \circ p$. Then note that

$$p_i^{-1} = \sum_n \frac{c_{ni}}{c_n} q_n^{-1}.$$

Since $f_i = p_i^{-1} \circ p$,

$$f_i(x) = p_i^{-1}(p(x)) = \sum_n \frac{c_{ni}}{c_n} q_n^{-1}(p(x)) = \sum_n \frac{c_{ni}}{c_n} g_n(x)$$

for every i and $x \in \boldsymbol{R}_{++}$. ///

An important corollary of this proposition is that if the economy is populated by consumers exhibiting constant relative risk aversion, then the marginal utility function for the representative consumer essentially uniquely determines the risk-sharing rules.

Corollary 1 *Let $I \in \{1, 2, \dots\}$ and $(p_1, \dots, p_I) \in \mathscr{Q}^I$, with the constant relative risk aversion $(\beta_1, \dots, \beta_I)$. Let $f = (f_1, \dots, f_I) \in \mathscr{F}^I$ and $p \in \mathscr{R}$ be derived from $(p_1, \dots, p_I)$ via FOC. Let p be written as (6) and $g \in \mathscr{F}^N$ be defined by (22). Assume that $s_n \neq s_m$ whenever $n \neq m$. Then:*

1. *$\{\beta_1, \dots, \beta_I\} = \{1/s_1, \dots, 1/s_N\}$.*
2. *For every $i = 1, \dots, I$, let $n = 1, \dots, N$ satisfy $\beta_i = 1/s_n$, then there exists a $\theta_i \in (0, 1]$ such that $f_i = \theta_i g_n$.*

Part 1 of this corollary claims that once we identify the pricing kernel, or the representative consumer's marginal utility function, we can completely recover all individual consumers' relative risk aversion. Part 2 then claims that their risk-sharing rules are determined uniquely up to scalar multiplication. This is in stark contrast in the case of general utility functions (Theorems 2 and 3), where the representative consumer's (marginal) utility function in no way restricts the shape of the efficient risk-sharing rules. Part 2 also establishes a mutual fund theorem among consumers with the same level of constant relative risk aversion: if consumers have the same level of constant relative risk aversion, then their risk-sharing rules are scalar multiples of the same g_n, and hence scalar multiples of one another.

Proof of Corollary 1 Let $p_i(x_i) = \lambda_i x_i^{-\beta_i}$ for each i. Since p is derived from $(p_1, \dots, p_I)$ via FOC,

$$p^{-1}(z) = \sum_{i=1}^{I} p_i^{-1}(z) = \sum_{i=1}^{I} \lambda_i^{1/\beta_i} z^{-1/\beta_i}. \tag{11}$$

Since this is equal to (6) for every $z \in \boldsymbol{R}_{++}$, it is easy to show by induction on N that $\{\beta_1, \dots, \beta_I\} = \{1/s_1, \dots, 1/s_N\}$, which establishes part 1, and

$$c_n = \sum_{\{i \mid \beta_i = 1/s_n\}} \lambda_i^{1/\beta_i} = \sum_{\{i \mid \beta_i = 1/s_n\}} \lambda_i^{s_n} \tag{12}$$

for every n. For each i and n, define $c_{ni} = \lambda_i^{s_n}/c_n \in (0,1]$ if $\beta_i = 1/s_n$ and $c_{ni} = 0$ otherwise. Then (8) hold for this choice of $(c_{1i}, \dots, c_{Ni})$ and part 2 follows from Proposition 3. ///

In the macroeconomics literature, the *aggregation property* often refers to the property that the equilibrium asset prices are not affected by any change in the wealth distribution among consumers as long as the mean is unaffected. The proof of Corollary 1 shows how the aggregation property fails in an economy populated by consumers with differing levels of constant relative risk aversion. In the individual consumers' marginal utility functions $p_i(x_i) = \lambda_i x_i^{-\beta_i}$, the coefficients λ_i roughly represent their wealth shares in the economy. If there is a transfer in wealth from one consumer to another of the same level of constant relative risk aversion, then the value of c_n in (12) is unchanged and the pricing kernel p is also unchanged, satisfying the aggregation property, as can be seen by (11). On the other hand, if the transfer is between two consumers of differing levels of constant relative risk aversion, then (12) shows that p is changed, violating the aggregation property. Given that the mutual fund theorem fails among consumers with differing levels of constant relative risk aversion, it is perhaps not surprising that the aggregation property fails as well. The surprising part of Corollary 1, however, is that any transfer between two consumers of differing levels of constant relative risk aversion must necessarily change the pricing kernel p (and hence the equilibrium asset prices) so that if p is identified, then the wealth distribution among consumers of differing levels of constant relative risk aversion can be uniquely determined.

For future references, we restate Proposition 3 and Corollary 1 in terms of utility functions.

Proposition 4 *Let $I \in \{1, 2, \dots\}$. Let $(u_1, \dots, u_I) \in \mathscr{W}^I$ be given by*

$$\left(u_i'\right)^{-1}(z) = \sum_n c_{ni} z^{-s_n}$$

for each i, and $f = (f_1, \dots, f_I) \in \mathscr{F}^I$ be derived from $(u_1, \dots, u_I)$ via WMP. For each n, write $c_n = \sum_i c_{ni}$ and define $g_n : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by

$$g_n^{-1}(z) = \sum_{m=1}^{N} \frac{c_m}{c_n^{s_m/s_n}} z^{s_m/s_n} \tag{13}$$

for every $z \in \boldsymbol{R}_{++}$. Then

$$f_i(x) = \sum_{n=1}^{N} \frac{c_{ni}}{c_n} g_n(x) \tag{14}$$

for every i and every $x \in \boldsymbol{R}_{++}$.

Corollary 2 *Let $I \in \{1, 2, \dots\}$ and $(u_1, \dots, u_I) \in \mathscr{V}^I$, with the constant relative risk aversion $(\beta_1, \dots, \beta_I)$. Let $f = (f_1, \dots, f_I) \in \mathscr{F}^I$ and $u \in \mathscr{W}$ be derived from $(u_1, \dots, u_I)$ via WMP. Write u as*

$$\left(u'\right)^{-1}(z) = \sum_n c_n z^{-s_n}, \tag{15}$$

where $s_n \neq s_m$ *whenever* $n \neq m$*, and define* $g = (g_1, \ldots, g_N) \in \mathscr{F}^N$ *by (22). Then:*

1. $\{\beta_1, \ldots, \beta_I\} = \{1/s_1, \ldots, 1/s_N\}$*.*

2. *For every* $i = 1, \ldots, I$*, let* $n = 1, \ldots, N$ *satisfy* $\beta_i = 1/s_n$*, then there exists a* $\theta_i \in (0, 1]$ *such that* $f_i(x) = \theta_i g_n(x)$ *for every* $x \in \boldsymbol{R}_{++}$*.*

In concluding this section, we give an application of Corollary 2. Suppose that an economy is populated by I consumers exhibiting constant relative risk aversion $(\beta_1, \ldots, \beta_I)$. Assume that it is not true that $\beta_1 = \cdots = \beta_I$. Let $f = (f_1, \ldots, f_I)$ be efficient risk-sharing rules and $\hat{u}$ be the utility function for a representative consumer in this economy. Then $(\hat{u}')^{-1}$ can be written as the right-hand side of (15). Since the relative risk aversion is nothing but the elasticity of $\hat{u}'$ multiplied by -1, Corollary 6 in the Appendix implies that it is strictly decreasing from $1/(\min_i 1/\beta_i) = \max \beta_i$ to $1/(\max_i 1/\beta_i) = \min \beta_i$. Denote the elasticity of f_i by $e_i : \boldsymbol{R}_{++} \to \boldsymbol{R}$, that is, $e_i(x) = f_i'(x)x/f_i(x)$. Thus, by Corollary 4, e_i is strictly decreasing, from $1/\min_j(\beta_i/\beta_j) = (\max_j \beta_j)/\beta_i$ to $1/\max_j(\beta_i/\beta_j) = (\min_j \beta_j)/\beta_i$. A closely related analysis was already given in Section 7 of HHK.

Now let $u \in \mathscr{U}$ exhibit constant relative risk aversion β, as would be concluded by the mutual fund theorem were all consumers to have equal constant relative risk aversion. By Theorem 2, there exists a $(u_1, \ldots, u_I) \in \mathscr{U}$ such that f and u are derived from $(u_1, \ldots, u_I)$ via WMP. It follows from (2) in Lemma 1 of HHK that

$$b_i(f_i(x)) = \frac{\beta}{e_i(x)}$$

where b_i is the relative risk aversion of u_i. Since e_i is strictly decreasing from $(\max_j \beta_j)/\beta_i$ to $(\min_j \beta_j)/\beta_i$, this shows that b_i is strictly increasing from $\beta\beta_i/(\max_j \beta_j)$ to $\beta\beta_i/(\min_j \beta_j)$.

The finding of this application of Corollary 2 can be paraphrased as follows. We take up an economy in which all consumers exhibit constant relative risk aversion, albeit at different levels. We find that the elasticities of the efficient risk-sharing rules are strictly decreasing and that the representative consumer exhibits strictly decreasing relative risk aversion. We then assume, contrary to this fact, that the representative consumer exhibits constant relative risk aversion, as is often done in macroeconomics. Then, to be consistent with these nonlinear risk-sharing rules, all consumers must exhibit strictly increasing relative risk aversion. A rough but easy reason for this goes as follows: The risk-sharing rules are nonlinear only if the individual consumers have different levels of relative risk aversion. According to Proposition 6 of HHK, this heterogeneity would then lead to strictly decreasing relative risk aversion for the representative consumer were they to exhibit nonincreasing relative risk aversion. Therefore, whenever we postulate that the representative consumer exhibits constant relative risk aversion in the presence of heterogeneous risk attitudes, in order to cancel out the tendency for strictly decreasing relative risk aversion arising from the heterogeneity, the individual consumers must necessarily exhibit strictly increasing relative risk aversion.

6 Identical Risk Attitudes

In this section, we consider what kind of restriction can be imposed on the efficient risk-sharing rules and the represetative consumer's utility function when all consumers have the same risk attitudes.

Let $(u_1, \ldots, u_I) \in \mathscr{U}^I$. Then, as is well known from expected utility theory, all consumers have the same risk attitudes if and only if the u_i are affine transformations of one another. Since the constant terms do not matter to the solution to WMP, we can assum without loss of generality that there are a $\hat{u} \in \mathscr{U}$ and a $(\lambda_1, \ldots, \lambda_I) \in \boldsymbol{R}^I_{++}$ such that $(u_1, \ldots, u_I) = (\lambda_1 \hat{u}, \ldots, \lambda_I \hat{u})$. It is then easy to check, using the first-order condition, that for any i and j, if $f_i(x) > f_j(x)$ for some $x \in \boldsymbol{R}_{++}$, then $f_i(x) > f_j(x)$ for every $x \in \boldsymbol{R}_{++}$. That is, for any two consumers, they always enjoy the same consumption level, or one always enjoys a higher consumption level than the other.[5]

Using a set of panel data of villages in India and Paskitan, Mazzocco and Saini (2006) showed that for most pairs of consumers (households), either one consumes more at some ggregate consumption level, but consumes less at another. This result implies that if the consumers have the time-additive expected utility functions with comon beliefs and if the allocations attained are efficient, then their risk attitudes must be different.

6.1 Two Consumers

We saw that for any two consumers, they always enjoy the same consumption level, or one always enjoys a higher consumption level than the other. We now show that this property is the only property that generally holds with identical risk attitudes in a two-consumer economy.

Theorem 4 *Let $f = (f_1, f_2) \in \mathscr{F}^2$ satisfy $f_1(x) < f_2(x)$ for every $x \in \boldsymbol{R}_{++}$, then there exists a $\hat{u} \in \mathscr{U}$ and $(\lambda_1, \lambda_2) \in \boldsymbol{R}^2_{++}$ such that $\lambda_1 < \lambda_2$ and f is derived from $(\lambda_1 \hat{u}, \lambda_2 \hat{u})$ via WMP.*

Proof of Theorem 4 Define $g : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $g = f_2 \circ f_1^{-1}$, then g is of C^1, $g(\boldsymbol{R}_{++}) = \boldsymbol{R}_{++}$, and $g'(\boldsymbol{R}_{++}) \subseteq \boldsymbol{R}_{++}$. Moreover, $g > \chi$. Then let $x_0 \in \boldsymbol{R}_{++}$ and, for each positive integer n, define inductively $x_n \in \boldsymbol{R}_{++}$ by letting $x_n = g(x_{n-1})$ and $x_{-n} = g^{-1}\left(x_{-(n-1)}\right)$. This is equivalent to letting $x_n = g(x_{n-1})$ for each (positive or negative) integer n. Then it is easy to check that x_n is strictly increasing in n, and that $x_n \to \infty$ and $x_{-n} \to 0$ as $n \to \infty$.

Let $h : [x_0, x_1) \to \boldsymbol{R}_{++}$ be any continuous function such that

$$\lim_{z \uparrow x_1} h(z) = \frac{h(x_0)}{g'(x_0)}.$$

Such an h in fact exists. For example, we can define

$$h(z) = \frac{\dfrac{1}{g'(x_0)} - 1}{x_1 - x_0}(z - x_0) + 1.$$

[5] On the other hand, if all consumers always enjoy the same consumption level, that is, $f_i = (1/I)\chi$ for every i, then $f = (f_1, \ldots, f_I)$ can be derived via WMP from the profile of any common utility function, $(\hat{u}, \ldots, \hat{u})$.

Then define $a : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $a(z) = h(z)$ for every $z \in [x_0, x_1)$, and then inductively

$$a(z) = \frac{a\left(g^{-1}(z)\right)}{g'\left(g^{-1}(z)\right)} \tag{16}$$

for every $z \in [x_n, x_{n+1})$ with $n \geq 1$ (where $g^{-1}(z) \in [x_{n-1}, x_n)$); and

$$a(z) = a\left(g(z)\right) g'(z) \tag{17}$$

for every $z \in \left[x_{-n}, x_{-(n-1)}\right)$ with $n \geq 1$ (where $g(z) \in \left[x_{-(n-1)}, x_{-(n-2)}\right)$). We prove that a is continuous. By the construction of a and the continuity of h and g', a is continuous on every open interval (x_n, x_{n+1}) and $\left(x_{-n}, x_{-(n-1)}\right)$. At x_0, every x_n, and every x_{-n}, a is continuous from right. It thus remains to prove that a is continuous from left at these points. We shall do this inductively.

For the continuity at x_0,

$$\lim_{z \uparrow x_0} a(z) = \lim_{z \uparrow x_0} \left(h(g(z))g'(z)\right) = \left(\lim_{z \uparrow x_1} h(z)\right) \left(\lim_{z \uparrow x_0} g'\left(z\right)\right) = \frac{h(x_0)}{g'(x_0)} g'(x_0) = a(x_0).$$

Hence a is continuous from left at x_0. Let $n \geq 1$. Suppose that a is continuous from left at x_{n-1}. Then

$$\lim_{z \uparrow x_n} a(z) = \lim_{z \uparrow x_n} \frac{a\left(g^{-1}(z)\right)}{g'\left(g^{-1}(z)\right)} = \lim_{z \uparrow x_{n-1}} \frac{a\left(z\right)}{g'\left(z\right)} = \frac{a\left(x_{n-1}\right)}{g'\left(x_{n-1}\right)} = a(x_n).$$

Thus a is continuous from left at x_n. Suppose that a is continuous from left at $x_{-(n-1)}$. Then

$$\lim_{z \uparrow x_{-n}} a(z) = \lim_{z \uparrow x_{-n}} \left(a\left(g(z)\right) g'(z)\right) = \left(\lim_{z \uparrow x_{-(n-1)}} a(z)\right) \left(\lim_{z \uparrow x_{-n}} g'(z)\right) = a\left(x_{-(n-1)}\right) g'\left(x_{-n}\right) = a(x_{-n}).$$

Thus a is continuous from left at x_{-n}. We can therefore conclude that $a : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ is continuous on the entire domain.

Define $\hat{u} : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by

$$\hat{u}(x) = \int_{x_0}^{x} \exp\left(-\int_{x_0}^{y} a(z)\, dz\right) dy. \tag{18}$$

Since a is continuous, $\hat{u}$ is of C^2. Moreover,

$$\begin{aligned} \hat{u}'(x) &= \exp\left(-\int_{x_0}^{x} a(z)\, dz\right) = \exp\left(\int_{x}^{x_0} a(z)\, dz\right), \\ \hat{u}''(x) &= -\hat{u}'(x)a(x). \end{aligned} \tag{19}$$

Thus $\hat{u}'(x) > 0 > \hat{u}''(x)$ for every $x \in \boldsymbol{R}_{++}$. To prove that $\hat{u} \in \mathscr{U}$, therefore, it suffices to show that $\hat{u}'(x) \to 0$ as $x \to \infty$ and $\hat{u}'(x) \to \infty$ as $x \to 0$. By integration by substitution, for every

$n \geq 1$,

$$\int_{x_n}^{x_{n+1}} a(z)\,dz = \int_{g(x_{n-1})}^{g(x_n)} \frac{a\left(g^{-1}(z)\right)}{g'\left(g^{-1}(z)\right)}\,dz = \int_{g(x_{n-1})}^{g(x_n)} a\left(g^{-1}(z)\right)\left(g^{-1}\right)'(z)\,dz = \int_{x_{n-1}}^{x_n} a(z)\,dz$$

$$\int_{x_{-n}}^{x_{-(n-1)}} a(z)\,dz = \int_{g^{-1}\left(x_{-(n-1)}\right)}^{g^{-1}\left(x_{-(n-2)}\right)} a\left(g(z)\right)g'(z)\,dz = \int_{x_{-(n-1)}}^{x_{-(n-2)}} a(z)\,dz.$$

Thus

$$\int_{x_0}^{x_n} a(z)\,dz = \int_{x_{-n}}^{x_0} a(z)\,dz = n\int_{x_0}^{x_1} a(z)\,dz.$$

Hence

$$\int_{x_0}^{x} a(z)\,dz \to \infty \text{ as } x \to \infty,$$
$$\int_{x}^{x_0} a(z)\,dz \to \infty \text{ as } x \to 0.$$

By (19) $\hat{u}'(x) \to 0$ as $x \to \infty$ and $\hat{u}'(x) \to \infty$ as $x \to 0$. Thus $\hat{u} \in \mathscr{U}$.

To complete the proof of this theorem, it suffices to show that there exists a $\mu > 1$ such that $\hat{u}'\left(f_1(x)\right) = \mu\hat{u}'\left(f_2(x)\right)$ for every $x \in \boldsymbol{R}_{++}$. This is equivalent to

$$\hat{u}'(x) = \mu\hat{u}'\left(g(x)\right) \tag{20}$$

for every $x \in \boldsymbol{R}_{++}$, which we shall prove.

Note first that the definition of (16) in fact implies that the equality of (17) holds for every $z \in \boldsymbol{R}_{++}$, regardless of whether it is larger or smaller than x_0. Thus, for every $x \in \boldsymbol{R}_{++}$,

$$\int_{x_0}^{x} a(z)\,dz = \int_{x_0}^{x} a(g(z))g'(z)\,dz = \int_{x_1}^{g(x)} a(z)\,dz.$$

Hence

$$\exp\left(-\int_{x_0}^{x} a(z)\,dz\right) = \exp\left(-\int_{x_1}^{g(x)} a(z)\,dz\right).$$

By (19),

$$\hat{u}'(x) = \hat{u}'\left(g(x)\right)\exp\left(\int_{x_0}^{x_1} a(z)\,dz\right).$$

Since $\exp\left(\int_{x_0}^{x_1} a(z)\,dz\right) > 1$, this completes the proof. ///

[Give restrictions on u.]

6.2 More Than Two Consumers

If there are more than two consumers in the economy, then there are more restrictions on the efficient risk-sharing rules than the condition that for any pair of two consumers, either the two always enjoy the same consumption level or one always consumes more than the other. We now present such a restriction. For a function g defined on $\boldsymbol{R}_{++}$ into itself and for a positive integer n, we denote by g^n the n-times composite function $\underbrace{g \circ \cdots \circ g}_{n \text{ times}}$.

Proposition 5 *Let $f = (f_1, \ldots, f_I) \in \mathscr{F}^I$ be derived from $(u_1, \ldots, u_I)$ via WMP for which the u_i are affine transformations of one another. Then, for all i and j, for all positive integers m and n, and for all $z \in \boldsymbol{R}_{++}$, if $\left(f_i \circ f_1^{-1}\right)^n(z) = \left(f_j \circ f_1^{-1}\right)^m(z)$, then $\left(\left(f_i \circ f_1^{-1}\right)^n\right)'(z) = \left(\left(f_j \circ f_1^{-1}\right)^m\right)'(z)$.*

Proof of Proposition 5 Let $\hat{u} \in \mathscr{U}$ be such that f is derived from $(\lambda_1 \hat{u}, \ldots, \lambda_I \hat{u})$ via WMP for some $(\lambda_1, \ldots, \lambda_I) \in \boldsymbol{R}^I_{++}$. Then let $a = -\hat{u}''/\hat{u}'$, the absolute risk aversion of $\hat{u}$. For each i, write g_i for $f_i \circ f_1^{-1}$. Then for every i and every $z \in \boldsymbol{R}_{++}$, $a(z) = a(g_i(z))g_i'(z)$. By an induction argument, we can show that for every i, every $z \in \boldsymbol{R}_{++}$, and every positive integer n, $a(z) = a\left(g_i^n(z)\right)\left(g_i^n\right)'(z)$. Thus, for all i and j, for all positive integers m and n, and for all $z \in \boldsymbol{R}_{++}$, $a\left(g_i^n(z)\right)\left(g_i^n\right)'(z) = a\left(g_j^m(z)\right)\left(g_j^m\right)'(z)$. The proposition then follows. ///

We now give an example of a risk-sharing rule of a three-consumer economy such that for any pair of two consumers, one always consumes more than the other and yet it cannot be efficient if they all have the same risk attitudes.

Example 1 For each $i = 1, 2, 3$, define $g_i : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by

$$\begin{aligned} g_1(z) &= z, \\ g_2(z) &= z + z^2, \\ g_3(z) &= z + 2z^2 + z^3 + 2z^4. \end{aligned}$$

Then $g_i' > 0$ and g_i is onto for every i. Thus g_i has the inverse function for every i. Moreover, the sum of the three functions, $g_1 + g_2 + g_3$, has the inverse function as well. So, for each i, define $f_i : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $f_i = g_i \circ (g_1 + g_2 + g_3)^{-1}$.

Proposition 6 *In Example 1,*

1. *$f = (f_1, f_2, f_3) \in \mathscr{F}^3$.*
2. *$f_1(x) < f_2(x) < f_3(x)$ for every $x \in \boldsymbol{R}_{++}$.*
3. *There is no (u_1, u_2, u_3) such that the u_i are affine transformations of one another and f is derived from (u_1, u_2, u_3) via WMP.*

Proof of Proposition 6 1. Since g_i is of C^1 and $g_i' > 0$, f_i is also of C^1 for every i. Since both g_i and $g_1 + g_2 + g_3$ have inverse functions, so does f_i and, in particular $f_i\left(\boldsymbol{R}_{++}\right) = \boldsymbol{R}_{++}$. Finally, $f_1 + f_2 + f_3 = (g_1 + g_2 + g_3) \circ (g_1 + g_2 + g_3)^{-1} = \chi$. Thus $f \in \mathscr{F}^3$.

2. This follows from $g_1(x) < g_2(x) < g_3(z)$ for every $z \in \boldsymbol{R}_{++}$.

3. Note first that $f_i \circ f_1^{-1} = g_i$ for every i. We prove that $g_2^2(1) = g_3(1)$ and yet $\left(g_2^2\right)'(1) \neq g_3'(1)$. By Proposition 5, this establishes part 3. For this purpose, note that

$$g_2^2(z) = \left(z + z^2\right) + \left(z + z^2\right)^2 = z + 2z^2 + 2z^3 + z^4$$

for every $z \in \boldsymbol{R}_{++}$. Thus $g_2^2(1) = 6 = g_3(1)$. On the other hand,

$$\left(g_2^2\right)'(z) = 1 + 4z + 6z^2 + 4z^3,$$
$$g_3'(z) = 1 + 4z + 3z^2 + 8z^3$$

for every $z \in \boldsymbol{R}_{++}$. Thus $\left(g_2^2\right)'(1) = 15 \neq 16 = g_3'(1)$. ///

7 General Domains

In this section we state, without proof, the generalizations of the concepts and results in the preceding sections to the case where the domain of the consumers' utility functions need not be $\boldsymbol{R}_{++}$. Although $\boldsymbol{R}_{++}$ is the most natural set of possible consumption levels, it is important to accommodate other sets as well. For example, if a utility function satisfies the Inada condition and also exhibits constant absolute risk aversion, its domain must necessarily be $\boldsymbol{R}$; if a utility function is quadratic and satisfies the Inada condition, its domain must necessarily be $(-\infty, d)$ for some $d < \infty$; if a utility function satisfies the Inada condition and also exhibits hyperbolic absolute risk aversion and strictly decreasing relative risk aversion, its domain must necessarily be (d, ∞) for some $d > 0$; and if a utility function satisfies the Inada condition and also exhibits hyperbolic absolute risk aversion and strictly increasing relative risk aversion, its domain must necessarily be (d, ∞) for some $d < 0$. The last two cases are especially of empirical relevance, as exemplified by Ogaki and Zhang (2001).

So we let $\mathscr{D}$ be the set of all open intervals in $\boldsymbol{R}$. For each $D \in \mathscr{D}$ and $r \in \{2, 3, \dots, \infty, \omega\}$, let $\mathscr{U}_{D,r}$ be the set of all C^r functions $u : D \to \boldsymbol{R}$ that satisfy such that $u'(D) = \boldsymbol{R}_{++}$ and $u''(D) \subseteq -\boldsymbol{R}_{++}$. For each $I \in \{1, 2, \dots\}$, $(D_1, \dots, D_I) \in \mathscr{D}^I$, and $r \in \{2, 3, \dots, \infty, \omega\}$, let $\mathscr{F}_r^{D_1 \times \cdots \times D_I}$ be the set of all C^r functions $f = (f_1, \dots, f_I) : \sum_i D_i \to D_1 \times \cdots \times D_I$, with $f_i : \sum_j D_j \to D_i$ for every i, such that $f_i\left(\sum_j D_j\right) = D_i$ and $f_i'\left(\sum_j D_j\right) \subseteq \boldsymbol{R}_{++}$ for every i, and $\sum_i f_i(x) = x$ for every $x \in \sum_i D_i$. To simplify exposition, we often write $\mathscr{U}_D$ for $\mathscr{U}_{D,2}$, and $\mathscr{F}^{D_1 \times \cdots \times D_I}$ for $\mathscr{F}_1^{D_1 \times \cdots \times D_I}$.

We say that $f : \sum_i D_i \to D_1 \times \cdots \times D_I$ and $u : \sum_i D_i \to \boldsymbol{R}$ are *derived from* $(u_1, \dots, u_I)$ *via the welfare maximization problem* (WMP for short) if $f(x)$ is a solution to (WMP) for every $x \in \boldsymbol{R}_{++}$ and u is the value function of (WMP), where the choice variables $(x_1, \dots, x_I)$ are now in $D_1 \times \cdots \times D_I$.

To avoid lengthy exposition, we shall present generalizations of some previoius results dealing with utility functions but not with marginal utility functions (pricing kernels). First, Theorems 1 and 2 can be generalized to the following ones.

Theorem 5 *For every* $I \in \{1, 2, \dots\}$, *every* $(D_1, \dots, D_I) \in \mathscr{D}^I$, *every* $r \in \{2, 3, \dots, \infty, \omega\}$, *and every* $(u_1, \dots, u_I) \in \mathscr{U}_{D_1,r} \times \cdots \times \mathscr{U}_{D_I,r}$*, if* $f : \sum_i D_i \to D_1 \times \cdots \times D_I$ *and* $u : \sum_i D_i \to \boldsymbol{R}$ *are derived from* $(u_1, \dots, u_I)$ *via WMP, then* $f \in \mathscr{F}_{r-1}^{D_1 \times \cdots \times D_I}$ *and* $u \in \mathscr{U}_{\sum_i D_i, r}$.

Theorem 6 *For every* $I \in \{1, 2, \dots\}$*, every* $(D_1, \dots, D_I) \in \mathscr{D}^I$*, every* $r \in \{2, 3, \dots, \infty, \omega\}$*, every* $f \in \mathscr{F}_{r-1}^{D_1 \times \cdots \times D_I}$*, and every* $u \in \mathscr{U}_{\sum_i D_i, r}$*, there exists a* $(u_1, \dots, u_I) \in \mathscr{U}_{D_1,r} \times \cdots \times \mathscr{U}_{D_I,r}$ *such that* f *and* u *are derived from* $(u_1, \dots, u_I)$ *via WMP. Moreover, for every* $(v_1, \dots, v_I) \in \mathscr{U}_{D_1,r} \times \cdots \times \mathscr{U}_{D_I,r}$*, if* $(v_1, \dots, v_I)$ *has the same property as this* $(u_1, \dots, u_I)$*, then* $u_i - v_i$ *is constant for every* i.

For a utility function $u \in \mathscr{U}_D$, the *absolute risk tolerance* $t : D \to \boldsymbol{R}_{++}$ is defined by $t(x) = -u'(x)/u''(x)$. This is nothing but the reciprocal of the Arrow-Pratt measure of absolute risk aversion. For each $d \in \boldsymbol{R}$, let $\mathscr{V}_{(d,\infty)}$ be the set of all $u \in \mathscr{U}_{(d,\infty)}$ for which there exists a $\kappa \in \boldsymbol{R}_{++}$ such that $t(x) = \kappa(x - d)$ for every $x > d$, where t is the absolute risk tolerance of u.[6] That is, $\mathscr{V}_{(d,\infty)}$ is the set of all utility functions of which the absolute risk tolerance is affine with a strictly positive slope κ and its value starts from zero at the minimum subsistence level d. This is equivalent to saying that every $u \in \bigcup_{d \in \boldsymbol{R}} \mathscr{V}_{(d,\infty)}$ exhibits hyperbolic absolute risk aversion. The value κ is often referred to as the *cautiousness*. If $d = 0$, then $\mathscr{V}_{(d,\infty)} = \mathscr{V}_{\boldsymbol{R}_{++}} = \mathscr{V}$ and the cautiousness is equal to the reciprocal of the value of constant relative risk aversion. If $d > 0$, then every $u \in \mathscr{V}_{(d,\infty)}$ exhibits strictly decreasing relative risk aversion. If $d < 0$, then every $u \in \mathscr{V}_{(d,\infty)}$ exhibits strictly increasing relative risk aversion.

It can be easily shown that $u \in \mathscr{V}_{(d,\infty)}$ if and only if there are a $\kappa \in \boldsymbol{R}_{++}$ and a $\lambda \in \boldsymbol{R}_{++}$ such that $u'(x) = \lambda(x-d)^{-1/\kappa}$ for every $x > d$. This is equivalent to saying that $(u')^{-1}(z) = \lambda^\kappa z^{-\kappa} + d$ for every $z \in \boldsymbol{R}_{++}$. We then let $\mathscr{W}_{(d,\infty)}$ be the set of all $u \in \mathscr{P}_{(d,\infty)}$ for which there exist an $N \in \{1, 2, \dots\}$, a $(c_1, \dots, c_N) \in \boldsymbol{R}_{++}^N$, and a $(s_1, \dots, s_N) \in \boldsymbol{R}_{++}^N$ such that

$$(u')^{-1}(z) = \sum_{n=1}^{N} c_n z^{-s_n} + d \tag{21}$$

for every $z \in \boldsymbol{R}_{++}$.

Proposition 7 *The set* $\bigcup_{d \in \boldsymbol{R}} \mathscr{W}_{(d,\infty)}$ *is the smallest subset of* $\bigcup_{d \in \boldsymbol{R}} \mathscr{U}_{(d,\infty)}$ *that includes* $\bigcup_{d \in \boldsymbol{R}} \mathscr{V}_{(d,\infty)}$ *and is closed under aggregation.*

Proposition 8 *Let* $I \in \{1, 2, \dots\}$ *and* $(d_1, \dots, d_I) \in \boldsymbol{R}^I$*. Let* $(u_1, \dots, u_I) \in \mathscr{W}_{(d_1,\infty)} \times \cdots \times \mathscr{W}_{(d_I,\infty)}$ *be given by*

$$(u_i')^{-1}(z) = \sum_n c_{ni} z^{-s_n} + d_i$$

for each i*, and* $f = (f_1, \dots, f_I) \in \mathscr{F}^{(d_1,\infty) \times \cdots \times (d_I,\infty)}$ *be derived from* $(u_1, \dots, u_I)$ *via WMP. For*

[6] It is not difficult to prove that for this affine absolute risk aversion t, the utility function u always satisfies the Inada condition.

each n, write $c_n = \sum_i c_{ni}$ and define $g_n : \left(\sum_j d_j, \infty\right) \to \left(\sum_{\{j|\kappa_j = s_n\}} d_j, \infty\right)$ by

$$g_n^{-1}(z) = \sum_{m=1}^{N} \frac{c_m}{c_n^{s_m/s_n}} \left(z - \sum_{\{j|\kappa_j = s_n\}} d_j \right)^{s_m/s_n} + \sum_j d_j \tag{22}$$

for every $z \in \boldsymbol{R}_{++}$. Then

$$f_i(x) = \sum_{n=1}^{N} \frac{c_{ni}}{c_n} \left(g_n(x) - \sum_{\{j|\kappa_j = s_n\}} d_j \right) + d_i \tag{23}$$

for every i and every $x \in \left(\sum_j d_j, \infty\right)$.

Corollary 3 *Let $I \in \{1, 2, \dots\}$ and $(d_1, \dots, d_I) \in \boldsymbol{R}^I$. Let $(u_1, \dots, u_I) \in \mathscr{V}_{(d_1,\infty)} \times \cdots \times \mathscr{V}_{(d_I,\infty)}$, with the constant cautiousness $(\kappa_1, \dots, \kappa_I)$. Let $f = (f_1, \dots, f_I) \in \mathscr{F}^{(d_1,\infty)\times\cdots\times(d_I,\infty)}$ and $u \in \mathscr{W}$ be derived from $(u_1, \dots, u_I)$ via WMP. Write u as*

$$\left(u'\right)^{-1}(z) = \sum_n c_n z^{-s_n} + \sum_j d_j,$$

where $s_n \neq s_m$ whenever $n \neq m$, and define $g = (g_1, \dots g_N) \in \mathscr{F}^N$ by (22). Then:

1. *$\{\kappa_1, \dots, \kappa_I\} = \{s_1, \dots, s_N\}$.*
2. *For every $i = 1, \dots, I$, let $n = 1, \dots, N$ satisfy $\kappa_i = s_n$, then there exists a $\theta_i \in (0, 1]$ such that $f_i(x) = \theta_i \left(g_n(x) - \sum_{\{j|\kappa_j = s_n\}} d_j\right) + d_i$ for every $x \in \left(\sum_j d_j, \infty\right)$.*

[Give an example.]

8 Conclusion

In this paper we have shown that the efficiency of risk allocation in no way restricts the nature of the risk-sharing rules beyond comonotonicity, or the nature of the pricing kernel beyond positivity and decreasingness. We have also explored implications of this result and investigated additional restrictions on the risk-sharing and pricing kernels when the individual consumers exhibit constant relative risk aversion, and when they have identical risk attitudes.

There are some unsolved problems. First, we should identify sufficient conditions on the efficient risk-sharing rules in economies with more than two consumers having identical risk attitudes. Such conditions would complement the necessary condition of Proposition 5. Second, we should extend the present analysis to the economy consisting of infinitely many consumers. It can be shown, for example, that if the (non-atomistic) individual consumers have utility functions of the form $u(x) = \left(x^{1-\beta} - 1\right) / (1 - \beta)$ with the constant relative risk aversion β distributed uniformly over an interval $\left[\underline{\beta}, \overline{\beta}\right]$, then the value function u of the maximization problem in which the summation in the objective function of (WMP) is replaced by an integral over $\left[\underline{\beta}, \overline{\beta}\right]$ satisfies

$$(u')^{-1}(z) = \frac{z^{-\underline{\beta}} - z^{-\overline{\beta}}}{\left(\overline{\beta} - \underline{\beta}\right) \log z}.$$

It is no longer a finite sum of power functions but is still an elementary function. This suggests that the functional form (6) defining the classes $\mathscr{R}$ and $\mathscr{W}$ may admit a tractable extension to the case of infinitely many consumers. Third, we should extend the present analysis to the case of multiple goods. A widely used setting in dynamic macroeconomics is where there are two goods, the consumption good and leisure. In the general case of L goods, the domains of the utility functions of (WMP) are $\boldsymbol{R}^L_{++}$ and the efficient risk-sharing rules are mappings from $\boldsymbol{R}^L_{++}$ to the I-tuples of $\boldsymbol{R}^L_{++}$. It is then necessary to accommodate non-additive utility functions to dientangle the degree of substitutabilities between the L goods and the degree of risk aversion across states. Theorem 2, however, cannot be extended to this case. More specifically, even in the case of two goods and two consumers, we can construct an example in which some non-additive utility function $u : \boldsymbol{R}^L_{++} \to \boldsymbol{R}$ for the representative consumer is compatible with some choices of efficient risk-sharing rules and but with others; and this is due to the the symmetry of the Hessian matrix $\nabla^2 u(x)$, which is tantamount to the integrability condition to demand theory. Attempts to extend Theorem 2 might stimulate a host of new directions of research.

A Appendix on the Sum of Power Functions

Proposition 9 *Let N be a positive integer, $(c_1, \dots, c_N) \in \boldsymbol{R}^N_{++}$, and $(s_1, \dots, s_N) \in \boldsymbol{R}^N_{++}$. Define $h : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $h(z) = \sum_{n=1}^N c_n z^{s_n}$ and its elasticity $e : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by $e(z) = h'(z)z/h(z)$.*

1. *The function h is strictly increasing and onto.*
2. *If $s_1 = \cdots = s_N$, then e is constant. Otherwise, it is strictly increasing. As $z \to \infty$, $e(z) \to \max_n s_n$. As $z \to 0$, $e(z) \to \min_n s_n$.*

Proof of Proposition 9 1. Since, for every n, $z \mapsto c_n z^{s_n}$ is strictly increasing, and $c_n z^{s_n} \to \infty$ as $z \to \infty$ and $c_n z^{s_n} \to 0$ as $z \to 0$, h is strictly increasing and onto.

2. Since

$$e(z) = \frac{\sum_{n=1}^N s_n c_n z^{s_n}}{\sum_{n=1}^N c_n z^{s_n}}, \tag{24}$$

if $s_1 = \cdots = s_N$, then $e(z) = s_1 = \cdots = s_N$ for every z. Otherwise, we prove the claim by induction on N.

Let's first prove for the case of $N = 2$. Without loss of generality, we can assume that $s_1 > s_2$. Then

$$e(z) = \frac{s_1 c_1 z^{s_1} + s_2 c_2 z^{s_2}}{c_1 z^{s_1} + c_2 z^{s_2}} = s_1 + \frac{(s_2 - s_1) c_2 z^{s_2}}{c_1 z^{s_1} + c_2 z^{s_2}} = s_1 - \frac{(s_1 - s_2) c_2}{c_1 z^{s_1 - s_2} + c_2}.$$

Since the far right hand side is strictly increasing, so is e. It also shows that $e(z) \to s_1 = \max_n s_n$ as $z \to \infty$, and

$$e(z) \to s_1 - \frac{(s_1 - s_2)c_2}{c_2} = s_2 = \min_n s_n$$

as $z \to 0$.

Let $N \geq 3$. We now suppose that the proposition is true for $N-1$ and then prove that it is true for N as well. We can assume without loss of generality that $s_i \geq s_N$ for every i. Then

$$\begin{aligned} e(z) &= \frac{\sum_{n=1}^{N-1} c_n z^{s_n}}{\sum_{n=1}^{N} c_n z^{s_n}} \frac{\sum_{n=1}^{N-1} s_n c_n z^{s_n}}{\sum_{n=1}^{N-1} c_n z^{s_n}} + \frac{s_N c_N z^{s_N}}{\sum_{n=1}^{N} c_n z^{s_n}} \\ &= \left(1 - \frac{c_N z^{s_N}}{\sum_{n=1}^{N} c_n z^{s_n}}\right) \frac{\sum_{n=1}^{N-1} s_n c_n z^{s_n}}{\sum_{n=1}^{N-1} c_n z^{s_n}} + \frac{c_N z^{s_N}}{\sum_{n=1}^{N} c_n z^{s_n}} s_N. \end{aligned} \tag{25}$$

Here the function

$$z \mapsto \frac{c_N z^{s_N}}{\sum_{n=1}^{N} c_n z^{s_n}} \tag{26}$$

is strictly decreasing because its derivative is

$$\begin{aligned} &\frac{c_N}{\left(\sum_{n=1}^{N} c_n z^{s_n}\right)^2} \left(s_N z^{s_N - 1} \left(\sum_{n=1}^{N} c_n z^{s_n} \right) - z^{s_N} \left(\sum_{n=1}^{N} s_n c_n z^{s_n - 1} \right) \right) \\ = &\frac{c_N}{\left(\sum_{n=1}^{N} c_n z^{s_n}\right)^2} \sum_{n=1}^{N} (s_N - s_n)\, c_n z^{s_N + s_n - 1} < 0. \end{aligned}$$

We now consider two separate cases. First, if $s_1 = \cdots = s_{N-1}$, then

$$\frac{\sum_{n=1}^{N-1} s_n c_n z^{s_n}}{\sum_{n=1}^{N-1} c_n z^{s_n}} = s_1 = \cdots = s_{N-1},$$

which is strictly greater than s_N. Since (26) is a strictly decreasing function, we see from (25) that e is a strictly increasing function.

Second, if it is not true that $s_1 = \cdots = s_{N-1}$, then, by the induction hypothesis,

$$z \mapsto \frac{\sum_{n=1}^{N-1} s_n c_n z^{s_n}}{\sum_{n=1}^{N-1} c_n z^{s_n}} \tag{27}$$

is a strictly increasing function, and

$$\frac{\sum_{n=1}^{N-1} s_n c_n z^{s_n}}{\sum_{n=1}^{N-1} c_n z^{s_n}} \to \min_{n \leq N-1} s_n$$

as $z \to 0$. Since $s_N \leq \min_{n \leq N-1} s_n$, the function (27) is always greater than s_N and strictly increasing. Since (26) is strictly decreasing, we conclude from (25) that e is strictly increasing.

The limit of $e(z)$ as $z \to \infty$ can be obtained by dividing both the denominator and the numerator of the right hand side of (24) by $z^{\max_i s_i}$. The limit of $e(z)$ as $z \to 0$ can be obtained by dividing both the denominator and the numerator of the right hand side of (24) by $z^{\min_i s_i}$. ///

We can use Proposition 9 to obtain similar results when the exponents are negative.

Corollary 4 *Let N be a positive integer, $(c_1, \dots, c_N) \in \boldsymbol{R}^N_{++}$, and $(s_1, \dots, s_N) \in \boldsymbol{R}^N_{++}$. Define $h : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $h(z) = \sum_{n=1}^N c_n z^{-s_n}$ and its elasticity, multiplied by -1, $e : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by $e(z) = -h'(z)z/h(z)$.*

1. *The function h is strictly decreasing and onto.*
2. *If $s_1 = \dots = s_N$, then e is constant. Otherwise, it is strictly decreasing. As $z \to \infty$, $e(z) \to \min_n s_n$. As $z \to 0$, $e(z) \to \max_n s_n$.*

Proof of Corollary 4 Define $\hat{h} : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $\hat{h}(z) = \sum_{n=1}^N c_n z^{s_n}$ and its elasticity $\hat{e} : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by $\hat{e}(z) = \hat{h}'(z)z/\hat{h}(z)$. Then $h(z) = \hat{h}(1/z)$ and $e(z) = \hat{e}(1/z)$. The claims of this corollary follow from Proposition 9. ///

Since the function h in Proposition 9 is strictly increasing and onto, it has an inverse function. Another corollary of Proposition 9 is concerned with the inverse function.

Corollary 5 *Let N be a positive integer, $(c_1, \dots, c_N) \in \boldsymbol{R}^N_{++}$, and $(s_1, \dots, s_N) \in \boldsymbol{R}^N_{++}$. Define $h : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $h(z) = \sum_{n=1}^N c_n z^{s_n}$, and denote its inverse function by $g : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$. Define the elasticity of g, $e : \boldsymbol{R}_{++} \to \boldsymbol{R}$, by $e(x) = g'(x)x/g(x)$.*

1. *The function g is strictly increasing and onto.*
2. *If $s_1 = \dots = s_N$, then e is constant. Otherwise, it is strictly decreasing. As $x \to \infty$, $e(x) \to 1/\max_n s_n$. As $x \to 0$, $e(x) \to 1/\min_n s_n$.*

Proof of Corollary 5 Part 1 follow from the fact that h is strictly increasing and onto. Define the elasticity of h, $\hat{e} : \boldsymbol{R}_{++} \to \boldsymbol{R}$, by $\hat{e}(z) = h'(z)z/h(z)$. Then $e(x) = (\hat{e}(g(x)))^{-1}$ for every $x \in \boldsymbol{R}_{++}$. Part 2 follows from this equality and Proposition 9. ///

The last corollary combines the cases covered by the preceding two corollaries.

Corollary 6 *Let N be a positive integer, $(c_1, \dots, c_N) \in \boldsymbol{R}^N_{++}$, and $(s_1, \dots, s_N) \in \boldsymbol{R}^N_{++}$. Define $h : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $h(z) = \sum_{n=1}^N c_n z^{-s_n}$, and denote its inverse function by $g : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$. Define the elasticity of g multiplied by -1, $e : \boldsymbol{R}_{++} \to \boldsymbol{R}$, by $e(x) = -g'(x)x/g(x)$.*

1. *The function g is strictly decreasing and onto.*
2. *If $s_1 = \dots = s_N$, then e is constant. Otherwise, it is strictly decreasing. As $x \to \infty$, $e(x) \to 1/\max_n s_n$. As $x \to 0$, $e(x) \to 1/\min_n s_n$.*

Proof of Corollary 6 Part 1 follow from the fact that h is strictly decreasing and onto. As for part 2, define the elasticity of h multiplied by -1, $\hat{e} : \boldsymbol{R}_{++} \to \boldsymbol{R}$, by $\hat{e}(z) = -h'(z)z/h(z)$. As shown in the proof of Corollary 5, $e(x) = (\hat{e}(g(x)))^{-1}$ for every $x \in \boldsymbol{R}_{++}$. Since $g(x) \to 0$ as $x \to \infty$ and $g(x) \to \infty$ as $x \to 0$, by Corollary 4, $\hat{e}(g(x)) \to \max_n s_n$ as $x \to \infty$ and $\hat{e}(g(x)) \to \min_n s_n$ as $x \to 0$. Thus $e(x) \to 1/\max_n s_n$ as $x \to \infty$ and $e(x) \to 1/\min_n s_n$ as $x \to 0$. ///

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